Métodos geométricos en problemas variacionales

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Miniworkshop: Órbitas en Análisis Matemático
Jerez de la Frontera, España

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Outline

- Introduction: variational problems.

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- Symmetries in variational problems.

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- Solvable pair of variational $\mathcal{C}^{\infty}$-symmetries.

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$$(x, u) \in M \subset \mathbb{R} \times \mathbb{R}, \quad M^{(n)} \text{ nth-order jet space } (x, u, u_1, \ldots, u_n)$$

$$u_i = \frac{d^i u}{dx^i}, \quad i = 1, \ldots, n.$$

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Find the functions $u = f(x)$ that maximize or minimize $\mathcal{L}[u]$.

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Warning

The determination of the space of admissible functions and the appropriated norms are quite delicate issues and lead to advanced topics on functional analysis. In our context we only consider C^∞ functions.

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Necessary condition: the Euler-Lagrange equation

$$E_u[L] = 0, \quad \text{where } E_u = \sum_{i=0}^{n} (-\mathrm{D}_x)^i \partial u_i,$$

being $D_x = \partial_x + u_1\partial_u + u_2\partial_{u_1} + \cdots$

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Find a curve of minimum lenght joinning two points (a, b) y (c, d) in the plane.

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$$\mathcal{L}[u] = \int_a^c \sqrt{1 + u_1^2} dx.$$

- The variational problem consists of minimizing $\mathcal{L}[u]$ over the space of differentiable functions such that $f(a) = b$ and $f(c) = d$.

Symmetries in variational problems

$$\mathcal{L}[u] = \int_\Omega L(x, u^{(n)})dx, \quad L(x, u^{(n)}) \text{ Lagrangian defined on } M^{(n)}.$$

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Definition 1 (Standard prolongation of a vector field)

The *prolongation of order n* of v is the vector field

$$\mathrm{v}^{(n)} = \xi(x, u)\partial_x + \sum_{i=0}^{n} \eta^{(i)}(x, u^{(i)})\partial_{u_i},$$

defined on $M^{(n)}$, where $\eta^{(0)}(x, u) = \eta(x, u)$ and, for $1 \leq i \leq n$,

$$\eta^{(i)}(x, u^{(i)}) \quad = \quad \mathrm{D}_x\left(\eta^{(i-1)}(x, u^{(i-1)})\right) - \mathrm{D}_x(\xi(x, u))u_i.$$

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$$\boxed{\mathrm{v} = \xi(x, u)\partial_x + \eta(x, u)\partial_u \in \mathfrak{X}(M)} \quad \text{v variational symmetry}$$

Definition 1

The vector field v is a variational symmetry (or Noether symmetry) if:

$$\mathrm{v}^{(n)}(L) + L\mathrm{D}_x\xi = 0.$$

P.J. Olver 1986.
Applications of Lie Groups to Differential Equations, Springer-Verlag, New York.

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Theorem 1

There exists a one-parameter family of variational problems $\widetilde{\mathcal{L}}_\mu[w]$, of order $n-1$, with Euler–Lagrange equation of order $2n-2$, such that the general solution of $E_u[L]=0$ can be found by a quadrature from the solutions of the Euler–Lagrange equations associated to $\widetilde{\mathcal{L}}_\mu[w]$, $\mu \in \mathbb{R}$.

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Theorem 1

Let $\mathrm{v} = \xi(x, u)\partial_x + \eta(x, u)\partial_u$ be a variational symmetry. Then the function

$$I = \frac{\partial L}{\partial u_1}(u_1\xi - \eta) - L\xi$$

is a first integral of the Euler-Lagrange equation $E_u[L] = 0$.

P.J. Olver 1986.

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Example: The Emden-Fowler equation

$$u_2 + \frac{2}{x} u_1 + u^5 = 0$$

- A suitable Lagrangian is $L(x, u, u_1) = x^2 \left(\frac{u_1^2}{2} - \frac{u^6}{6} \right)$.
- $\mathrm{v} = x\partial_x - \frac{1}{2} u \partial_u$ is a variational symmetry.
- First integral:

$$I = x^3 \frac{u^6}{6} + x^3 \frac{u_1^2}{2} + x^2 \frac{u u_1}{2}.$$

- Order reduction:

$$x^3 \frac{u^6}{6} + x^3 \frac{u_1^2}{2} + x^2 \frac{u u_1}{2} = C_1.$$

Variational $\mathcal{C}^\infty$-symmetries

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Definition 2 (λ-prolongation of vector fields)

The infinite λ-*prolongation* of v is the vector field

$$\mathrm{v}^\lambda = \xi(x,u)\partial_x + \sum_{i=0}^{+\infty} \eta^{[\lambda,(i)]}(x, u^{(i)})\partial_{u_i},$$

defined on $M^{(\infty)}$, where $\eta^{[\lambda,(0)]}(x,u) = \eta(x,u)$ and, for $1 \leq i$,

$$\eta^{[\lambda,(i)]}(x, u^{(i)}) = (\mathrm{D}_x + \lambda)\left(\eta^{[\lambda,(i-1)]}(x, u^{(i-1)})\right) - (\mathrm{D}_x + \lambda)(\xi(x,u))u_i.$$

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$$\boxed{\mathrm{v} = \xi(x, u)\partial_x + \eta(x, u)\partial_u \quad \lambda \in \mathcal{C}^\infty(M^{(1)})}$$

Definition 2

The pair (v, λ) is a variational $\mathcal{C}^\infty$-symmetry (or variational λ-symmetry) if:

$$\mathrm{v}^\lambda(L) + L(\mathrm{D}_x + \lambda)\xi = 0.$$

Muriel, C., Romero, J. L. and Olver, P.J. 2006.
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$\mathrm{v} = \xi(x,u)\partial_x + \eta(x,u)\partial_u \quad \lambda \in \mathcal{C}^\infty(M^{(1)})$ (v, λ) variational $\mathcal{C}^\infty$-symmetry

Theorem 2

There exists a variational problem $\widetilde{\mathcal{L}}[w] = \int \widetilde{L}(y, w^{(n-1)}) dy$, of order $n-1$, with Euler–Lagrange equation $E_w[\widetilde{L}] = 0$ of order $2n-2$, such that a $(2n-1)$-parameter family of solutions of $E_u[L] = 0$ can be found by solving a first order equation from the solutions of the reduced Euler–Lagrange equation $E_w[\widetilde{L}] = 0$.

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Example I

Let us consider the first-order variational problem

$$\mathcal{L}[u] = \int L(x, u, u_1)dx$$

with Lagrangian

$$L(x, u, u_1) = \left(u_1 - \frac{u}{x} + u^2 + 1\right)^2. \qquad (1)$$

- Euler-Lagrange equation:

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- The equation does not admit standard Lie symmetries, which implies that there not exist variational symmetries neither.

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- 1-parameter family of exact solutions:

$$u(x) = \frac{C\mathrm{J}_0(x) + \mathrm{Y}_0(x)}{C\mathrm{J}_1(x) + \mathrm{Y}_1(x)}, \qquad C \in \mathbb{R}. \tag{3}$$

Example II

Let us consider the first-order variational problem

$$\mathcal{L}[u] = \int L(x, u, u_1)dx$$

with Lagrangian

$$L(x, u, u_1) = \frac{1}{2}\left(u_1 - \frac{u}{x} + x^2\right)^2. \qquad (4)$$

- Euler-Lagrange equation:

$$u_2 = -\frac{3}{x}u^2 + 2u^3 = 0.$$

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- Euler-Lagrange equation:

$$u_2 = -\frac{3}{x}u^2 + 2u^3 = 0.$$

- The equation admits $\mathrm{v} = x\partial_x - u\partial_u$ as Lie point symmetry, which can be used to reduce the order.

Example II

- By means of the change

$$z = xu, \qquad h = \frac{-1}{x(u_1 x + u)}, \tag{5}$$

the equation reduces to $h_1 = (-2z^3 + 3z^2 + 2z)h^3 + 3h^2$.

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$$u(x) = \frac{2x}{x^2 + C}, \qquad C \in \mathbb{R}.$$

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- One-parameter family of exact solutions:
$$u(x) = \frac{2x}{x^2 + C}, \qquad C \in \mathbb{R}.$$

- General solution to the previous Abel equation in parametric form:
$$z = \frac{2x^2}{x^2 + C}, \qquad h = \frac{-(x^2 + C)^2}{4Cx^2}, \quad C \in \mathbb{R}.$$

Solvable pair of variational $\mathcal{C}^\infty$-symmetries

$$\mathcal{L}[u] = \int_\Omega L(x, u^{(n)})dx, \quad E_u[L] = 0.$$

$$\boxed{(\mathrm{v}_1, \lambda_1) \quad \text{and} \quad (\mathrm{v}_2, \lambda_2) \quad \text{variational } \mathcal{C}^\infty\text{-symmetries}}$$

Solvable pair of variational $\mathcal{C}^\infty$-symmetries

$$\mathcal{L}[u] = \int_\Omega L(x, u^{(n)})dx, \quad E_u[L] = 0.$$

$(\mathrm{v}_1, \lambda_1)$ and $(\mathrm{v}_2, \lambda_2)$ variational $\mathcal{C}^\infty$-symmetries

Definition 3

$(\mathrm{v}_1, \lambda_1)$ and $(\mathrm{v}_2, \lambda_2)$ form a solvable pair of variational $\mathcal{C}^\infty$-symmetries if:

- $\mathrm{v}_1^{\lambda_1}$ and $\mathrm{v}_2^{\lambda_2}$ pointwise linearly independent on M^∞.
- $\left[\mathrm{v}_1^{\lambda_1}, \mathrm{v}_2^{\lambda_2} \right] = h\, \mathrm{v}_1^{\lambda_1}$, for some function $h \in \mathcal{C}^\infty(M)$.

Solvable pair of variational $\mathcal{C}^\infty$-symmetries

$$\mathcal{L}[u] = \int_\Omega L(x, u^{(n)})dx, \quad E_u[L] = 0.$$

$(\mathrm{v}_1, \lambda_1)$ and $(\mathrm{v}_2, \lambda_2)$ solvable pair of variational $\mathcal{C}^\infty$-symmetries

Theorem 3

There exists a variational problem $\widehat{\mathcal{L}}[z] = \int \widehat{L}(x, z^{(n-2)})dx$ of order $n-2$ such that a $(2n-2)$-parameter family of solutions of $E_u[L] = 0$ can be reconstructed from the solutions of the associated $(2n-4)$-th order Euler–Lagrange equation $E_z[\widehat{L}] = 0$ by solving two successive first order ordinary differential equations.

Ruiz, A., Muriel, C. and Olver, P.J. 2018.
On the commutator of $\mathcal{C}^\infty$-symmetries and the reduction of Euler–Lagrange equations.
J. Phys. A: Math. Theor. **51** 145202-145223

Example

$$\mathcal{L}[u] = \int_\Omega L(x, u^{(2)})dx, \quad L(x, u^{(2)}) = \frac{x(2uu_2 - u_1^2 + 4u_1u^2 + u^4 + 1)^2}{4u^4}.$$

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$$\begin{aligned} -x - 2u^3 - 5u_1u - 2u^7 + 4u_3u^3 - 10u_1u^2u_2 - 5xu^5u_2 - 6xu^2u_2^2 \\ + 10xu_1^2 - 5u^5u_1 + 4u^4u_2 - 2u_1^2u^3 + 5u_1^3u + xu^8 \\ - 9xu_1^4 - 5xu_2u + 2xu_4u^3 + 21xuu_1^2u_2 - 8xu_3u^2u_1 = 0. \end{aligned}$$

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The Euler–Lagrange equation does not admit Lie point symmetries, which implies that the variational problem does not possess standard variational symmetries.

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Solvable pair of variational $\mathcal{C}^\infty$-symmetries

$$\left(\partial_u, \frac{u_1}{u} - u + \frac{1}{u}\right), \quad \left(u^2\partial_u, -\frac{u_1}{u} - u - \frac{1}{u}\right)$$

Example

First order reduction using $\left(\partial_u, \frac{u_1}{u} - u + \frac{1}{u}\right)$

Second-order invariants for $\left(\partial_u\right)^{\lambda_1}$ are

$$w = \frac{u_1 + 1 + u^2}{u}, \;\; w_1 = \frac{u_2 u - u_1^2 - u_1 + u_1 u^2}{u^2}. \tag{6}$$

Reduced Lagrangian:

$$\widetilde{L}(x, w, w_1) = \frac{x(w^2 + 2w_1 - 2)^2}{4}. \tag{7}$$

Euler-Lagrange equation

$$-2xw_2 - 2w_1 + (w^2 - 2)(xw - 1) = 0. \tag{8}$$

Example

Inherited variational $\mathcal{C}^\infty$-symmetry: $(-2\partial_w, -w)$

First-order invariant:

$$z = \frac{w^2}{2} + w_1. \tag{6}$$

Reduced Lagrangian:

$$\widehat{L}(x, z) = \frac{1}{4}x(-2 + 2z)^2, \tag{7}$$

whose Euler-Lagrange equation becomes

$$x(-2 + 2z) = 0. \tag{8}$$

Example

Solution: z=1

First reconstruction:

$$\frac{w^2}{2} + w_1 = 1,$$

which yields

$$w = \sqrt{2}\tanh\left(\frac{\sqrt{2}}{2}(x + C_1)\right), \tag{6}$$

where $C_1 \in \mathbb{R}$.
Second reconstruction:

$$\frac{u_1 + 1 + u^2}{u} = \sqrt{2}\tanh\left(\frac{\sqrt{2}}{2}(x + C_1)\right).$$

Example

Two-parameter family of solutions

$$u(x; C_1, C_2) = \frac{\sqrt{2}\,P(x)\left((C_2 - 1)\cos\left(\frac{\sqrt{2}}{2}x\right) + (C_2+1)\sin\left(\frac{\sqrt{2}}{2}x\right)\right)}{2\,Q(x)\cos\left(\frac{\sqrt{2}}{2}x\right) + 2\left(C_1 C_2 e^{\frac{\sqrt{2}}{2}x} + e^{-\frac{\sqrt{2}}{2}x}\right)\sin\left(\frac{\sqrt{2}}{2}x\right)},$$

where

$$P(x) = C_1 e^{\frac{\sqrt{2}}{2}x} - e^{\frac{-\sqrt{2}}{2}x} \quad \text{and} \quad Q(x) = C_2 e^{\frac{-\sqrt{2}}{2}x} - C_1 e^{\frac{\sqrt{2}}{2}x},$$

and $C_1, C_2 \in \mathbb{R}$.

Thank you for your attention