Stabilization for a class of three order switched linear systems

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Miniworkshop Órbitas en Análisis Matemático
Jerez de la Frontera
15 y 16 de octubre 2021

Outline of this presentation

Outline of this section

1. **Introduction to control theory**
   - Control system
   - Objective in control theory
   - Types of control systems
   - Types of controls

Control system

Control system

In a control system can be several types of variables:

- **time** variable

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In a control system can be several types of variables:

- **time** variable
- **state** variables

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- **control** variables

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- **control** variables
- **measurable** variables

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- **control** variables
- **measurable** variables
- **noise/uncertain** variables

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- **control** variables
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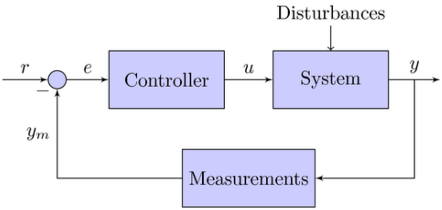

Objective in control theory

Objective

Design a control signal which achieves some "property" or feature desired for the state variable for every noise/uncertain signal.

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Design a control signal which achieves some “property” or feature desired for the state variable for every noise/uncertain signal.

To control the state variable

Objective in control theory

Properties for control problems

It can be studied different properties:

- **Stability**

Objective in control theory

Properties for control problems

It can be studied different properties:

- **Stability**
- **Stabilization**

Objective in control theory

Properties for control problems

It can be studied different properties:

- **Stability**
- **Stabilization**
- **Controlability/reachability**

Objective in control theory

Properties for control problems

It can be studied different properties:

- **Stability**
- **Stabilization**
- **Controlability/reachability**
- **Optimization**

Types of control systems

Classification of control systems

- **Discrete systems**: variables are discrete

Types of control systems

Classification of control systems

- **Discrete systems**: variables are discrete
- **Continuous systems**: variables are continuous

Types of control systems

Classification of control systems

- **Discrete systems**: variables are discrete
- **Continuous systems**: variables are continuous
- **Hybrid systems**: variables are discrete and continuous

Types of control systems

Type of dynamical system

- **ODE** (ordinary differential equation)

Types of control systems

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- **ODE** (ordinary differential equation)
- **PDE** (partial differential equation)

Types of control systems

Type of dynamical system

- **ODE** (ordinary differential equation)
- **PDE** (partial differential equation)
- **Difference equation**

Types of control systems

Type of dynamical system

- **ODE** (ordinary differential equation)
- **PDE** (partial differential equation)
- **Difference equation**
- A mixture: for instance, a **coupled ODE-PDE** system

Types of controls

Classification of control

- **Open loop control**: control depends on time

Types of controls

Classification of control

- **Open loop control**: control depends on time
- **Closed loop control (feedback)**: control depends on state. (Robust).

Outline of this section

Definition of switched system

Switched system

A **switched system** is given by a family of systems

$$\dot{x} = f_\sigma(x),$$

where $f_j : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field with index $j \in J$ which is called **subsystem**, the variable x is the **state variable** and the number n is the **order** of the system.

$\sigma : \mathbb{R}_+ \to J$ is a switching law.

$$\dot{x}(t) = f_{\sigma(t)}(x(t)).$$

$\sigma : \mathbb{R}^n \to J$ is a feedback switching law.

$$\dot{x}(t) = f_{\sigma(x(t))}(x(t)).$$

Definition of switched system

Switched linear system

A **switched linear system** is given by a family of linear systems

$$\dot{x} = A_\sigma(x),$$

where $A_1, \ldots, A_M \in \mathcal{M}_n$ are matrices.

Solution of a switched system

$$\begin{cases} \dot{x}(t) = f_{\sigma(t)}(x(t)), \ t \geq 0, \\ x(0) = x_0, \end{cases}$$

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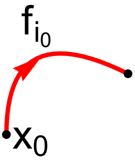

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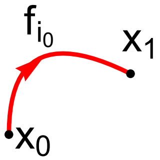

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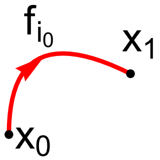

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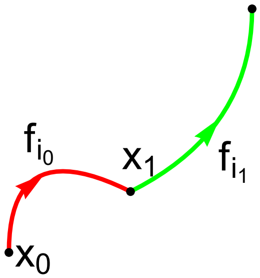

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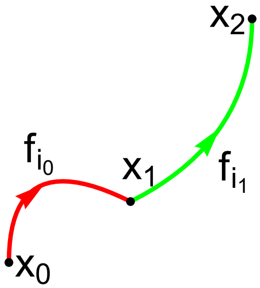

Solution of a switched system

$$\dot{x} = A_1 x = \begin{pmatrix} -2 & 1 \\ -2 & 3 \end{pmatrix} x, \qquad \dot{x} = A_2 x = \begin{pmatrix} -1/2 & -1 \\ 10 & -2 \end{pmatrix} x.$$

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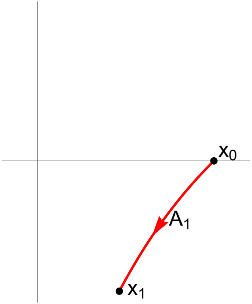

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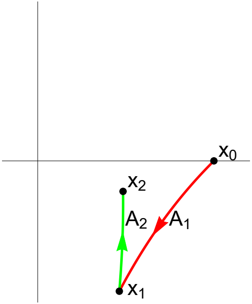

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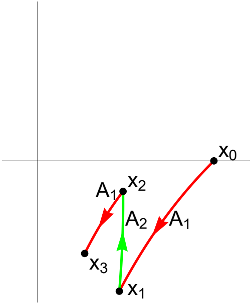

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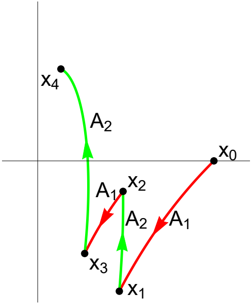

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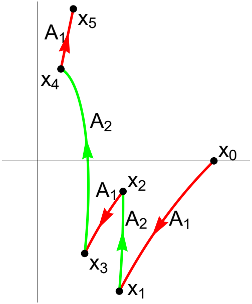

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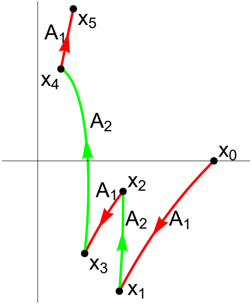

Stabilizability of switched systems

Exponentially stable

A dynamical system given by a differential equation

$$\dot{x}(t) = f(x),$$
$$x(0) = x_0,$$

is exponentially stable if there exist $C > 0$ and $\alpha > 0$ such that

$$\|\varphi(t; x_0)\| \leq Ce^{-\alpha t}\|x_0\|$$

for every $t \geq 0$ and $x_0 \in \mathbb{R}^n$.

Stabilizability of switched systems

The problem of stabilizability

For a switched system

$$\dot{x} = f_\sigma(x),$$

the problem is **to design a switching law** σ such that the dynamical system

$$\dot{x}(t) = f_{\sigma(t)}(x),$$
$$x(0) = x_0,$$

is exponentially stable.

Outline of this section

3. **Stabilization for a class of third order switched linear systems**
   - The class of third order switched linear systems
   - Previous results
   - The new class of third order switched linear systems
   - Main result
   - Parametrization of the class of switched linear systems
   - Numerical example

The class of third order switched linear systems

Switched system

$$\mathcal{S} = \{A_1, A_2, A_3\}$$

where $A_1, A_2, A_3 \in \mathcal{M}_3$ and

The class of third order switched linear systems

Switched system

$$\mathcal{S} = \{A_1, A_2, A_3\}$$

where $A_1, A_2, A_3 \in \mathcal{M}_3$ and

1. For $k = 1, 2, 3$, the eigenvalues of A_k are $\lambda_k, a_k \pm b_k i$, where $\lambda_k, a_k, b_k \in \mathbb{R}$ and $b_k > 0$.

The class of third order switched linear systems

Switched system

$$\mathcal{S} = \{A_1, A_2, A_3\}$$

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1. For $k = 1, 2, 3$, the eigenvalues of A_k are $\lambda_k, a_k \pm b_k i$, where $\lambda_k, a_k, b_k \in \mathbb{R}$ and $b_k > 0$.
2. If $v_1, v_2, v_3 \in \mathbb{R}^3$ are eigenvectors of matrices A_1, A_2, A_3 associated to the real eigenvalues $\lambda_1, \lambda_2, \lambda_3$, respectively, then v_1, v_2, v_3 are linearly independent.

Previous results

Lemma

Let $\mathcal{S} = \{A_1, \ldots, A_M\}$ be a switched linear system and $P \in \mathcal{M}_n$ be a non-singular matrix. The following statements are equivalent.

Previous results

Lemma

Let $\mathcal{S} = \{A_1, \ldots, A_M\}$ be a switched linear system and $P \in \mathcal{M}_n$ be a non-singular matrix. The following statements are equivalent.

1. The switched system $\mathcal{S}$ is (asymptotically, exponentially) stabilizable.

Previous results

Lemma

Let $\mathcal{S} = \{A_1, \ldots, A_M\}$ be a switched linear system and $P \in \mathcal{M}_n$ be a non-singular matrix. The following statements are equivalent.

1. The switched system $\mathcal{S}$ is (asymptotically, exponentially) stabilizable.
2. The switched system $\mathcal{S}' = \{PA_1P^{-1}, \ldots, PA_MP^{-1}\}$ is (asymptotically, exponentially) stabilizable.

Previous results

$$\mathcal{S} = \{A_1, A_2, A_3\}$$

1. For $k = 1, 2, 3$, the eigenvalues of A_k are $\lambda_k, a_k \pm b_k i$, where $\lambda_k, a_k, b_k \in \mathbb{R}$ and $b_k > 0$.
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$$P^{-1} = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$$

Previous results

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Lemma

$$\tilde{a}^1_{23}\tilde{a}^1_{32} < 0,\ \tilde{a}^2_{13}\tilde{a}^2_{31} < 0,\ \tilde{a}^3_{12}\tilde{a}^3_{21} < 0.$$

Previous results

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Previous results

Case $\tilde{a}^1_{23}\tilde{a}^2_{31}\tilde{a}^3_{12} > 0$

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Case $\tilde{a}^1_{23}\tilde{a}^2_{31}\tilde{a}^3_{12} > 0$

$$\tilde{\tilde{S}} = \{\tilde{\tilde{A}}_1, \tilde{\tilde{A}}_1, \tilde{\tilde{A}}_1\}$$

$$\tilde{\tilde{A}}_1 = Q\tilde{A}_1 Q^{-1},\ \tilde{\tilde{A}}_2 = Q\tilde{A}_2 Q^{-1},\ \tilde{\tilde{A}}_3 = Q\tilde{A}_3 Q^{-1}.$$

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$\tilde{a}^1_{23}$	$\tilde{a}^2_{31}$	$\tilde{a}^3_{12}$	s_1	s_2	s_3	$\tilde{\tilde{a}}^1_{23}$	$\tilde{\tilde{a}}^2_{31}$	$\tilde{\tilde{a}}^3_{12}$
$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$+$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$
$-$	$+$	$-$	$+$	$-$	$+$	$+$	$+$	$+$
$-$	$-$	$+$	$+$	$+$	$-$	$+$	$+$	$+$

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$\tilde{a}^1_{23}$	$\tilde{a}^2_{31}$	$\tilde{a}^3_{12}$	$\tilde{a}^1_{32}$	$\tilde{a}^2_{13}$	$\tilde{a}^3_{21}$	s_1	s_2	s_3	$\tilde{\tilde{a}}^1_{23}$	$\tilde{\tilde{a}}^2_{31}$	$\tilde{\tilde{a}}^3_{12}$
+	+	−	−	−	+	+	+	−	+	+	+
+	−	+	−	+	−	+	−	+	+	+	+
−	+	+	+	−	−	−	+	+	+	+	+
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3. The entries $\tilde{\tilde{a}}^1_{23}, \tilde{\tilde{a}}^2_{31}, \tilde{\tilde{a}}^3_{12}$ are positive.

The new class of third order switched linear systems

$$S = \{A_1, A_2, A_3\}$$

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1. the matrix A_k has eigenvalues $\lambda_k, a_k + b_k i, a_k - b_k i$ with $\lambda_k, a_k, b_k \in \mathbb{R}$ and $b_k > 0$, for $k = 1, 2, 3$,
2. $a^1_{23} > 0$, $a^2_{31} > 0$ and $a^3_{12} > 0$.

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Lemma

For an initial condition $x_0 \in \mathbb{R}^3 \setminus \{0\}$.

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Proposition

It turns out that

1. $e^{A_1 T_1}x_0 \in F_2$ for all $x_0 \in F_1$,
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Proof

For a given $x_0 \in F_1$ it follows that $x_0 = x_1 e_1 + x_3 e_3$ for $x_1, x_3 \in \mathbb{R}$. Then

$$e^{A_1 T_1} x_0 = x_1 e^{A_1 T_1} e_1 + x_3 e^{A_1 T_1} e_3 = x_1 e^{\lambda_1 T_1} e_1 + x_3 e^{A_1 T_1} e_3,$$

since $e_1 \in F_2$ and $e^{A_1 T_1} e_3 \in F_2$ then $e^{A_1 T_1} x_0 \in F_2$.

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For $x_0 \in \mathbb{R}^3 \setminus \{0\}$ and $t \geq 0$

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$$\alpha = -\frac{\log(\beta)}{T} > 0$$

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If $\rho(M) < 1$ then the switched linear system S is exponentially stable with switching law σ.

Proof

Since $\rho(M) < 1$ then the discrete-time linear system with matrix M is exponentially stable, i.e. there exist $c > 0$ and $\beta \in [0, 1)$ such that

$$\|M^k y\| \le c\beta^k \|y\| \quad \text{for } y \in \mathbb{R}^2 \text{ and } k \in \mathbb{N}$$

$$\|E^k x\| \le c\beta^k \|x\| \quad \text{for } x \in F_1 \text{ and } k \in \mathbb{N}$$

$$\alpha = -\frac{\log(\beta)}{T} > 0$$

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Proof (Cont.)

Let $x_0 \in \mathbb{R}^3 \setminus \{0\}$ be an initial condition.

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Let $x_0 \in \mathbb{R}^3 \setminus \{0\}$ be an initial condition.

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where $x_{k+1} = E^k e^{A_3\tau_3}x_0$, $k \in \mathbb{N}$ and $t \geq 0$.

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$$\|\varphi(t; x_0, \sigma)\| = \|e^{A_1\tau}x_{k+1}\| \leq m_2\|E^k e^{A_3\tau_3}x_0\|$$

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$$\|\varphi(t;x_0,\sigma)\| \le Ce^{-\alpha t}\|x_0\| \quad \text{for all } t \ge 0.$$

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Proposition

A polynomial $p(z) = z^2 + a_1 z + a_0$ is Schur stable (all the root are in the unit disk) if, and only if

$$|a_0| < 1 \quad \text{and} \quad |a_1| < a_0 + 1.$$

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Theorem

The switched linear system S is exponentially stable with switching law σ if the matrix M verifies the following conditions

1. $|\det(M)| < 1$, and
2. $|\operatorname{tr}(M)| < \det(M) + 1$.

Parametrization of the class of switched linear systems

Let $A \in \mathcal{M}_3$ be a matrix verifying the following conditions

1. A has eigenvalues $\lambda, a+bi, a-bi$ with $\lambda, a, b \in \mathbb{R}$ and $b > 0$,
2. the matrix A is

$$\begin{pmatrix} \lambda & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$$

with $a_{23} > 0$.

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$$J = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & a & b \\ 0 & -b & a \end{pmatrix} \text{ and } P = \begin{pmatrix} 1 & p & q \\ 0 & \gamma & 0 \\ 0 & -d & b \end{pmatrix},$$

where $b, \lambda > 0$

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(a - \lambda)p - bq &= a_{23}a_{12} - da_{13}, \\
bp + (a - \lambda)q &= ba_{13}.
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Exponential matrix e^{AT}

$$e^{AT} = Pe^{JT}P^{-1} = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ 0 & e_{22} & e_{23} \\ 0 & e_{32} & 0 \end{pmatrix}.$$

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Therefore

$$T = \frac{1}{b}\operatorname{arccot}\left(\frac{d}{b}\right),$$

$$\sin(bT) = \frac{b}{\sqrt{b^2+d^2}}, \text{ and } \cos(bT) = \frac{d}{\sqrt{b^2+d^2}}.$$

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$$e_{23} = \frac{\gamma}{\sqrt{b^2+d^2}} e^{aT}.$$

Parametrization of the class of switched linear systems

Matrices A_1, A_2 and A_3

$$A_1 = Q_1 A Q_1^{-1},\ A_2 = Q_2 A Q_2^{-1},\ A_3 = Q_3 A Q_3^{-1},$$

where

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ Q_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ Q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

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Matrix $E = e^{A_3 T_3} e^{A_2 T_2} e^{A_1 T_1} = E_3 E_2 E_1$

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Parametrization of the class of switched linear systems

tr(M)

$$|\operatorname{tr}(M)| < 1 + \det(M),$$

$$\operatorname{tr}(M) = e^3_{32} e^2_{21} e^1_{13} + e^3_{33} e^2_{31} e^1_{13} + e^3_{12} e^2_{21} e^1_{11} + e^3_{32} e^2_{22} e^1_{23},$$

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Numerical example

Consider a third order switched system with matrices

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2.5 \\ 0 & -20 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 0 & -12 \\ 0 & 1 & 0 \\ .5 & 0 & 1 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 1 & 0 \\ -12 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Numerical example

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$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2.5 \\ 0 & -20 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 0 & -12 \\ 0 & 1 & 0 \\ .5 & 0 & 1 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 1 & 0 \\ -12 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- Eigenvalues of A_1: $1, 1 \pm 5\sqrt{2}i$.
- Eigenvalues of A_2: $1, 1 \pm \sqrt{6}i$.
- Eigenvalues of A_3: $1, 1 \pm 2\sqrt{3}i$.

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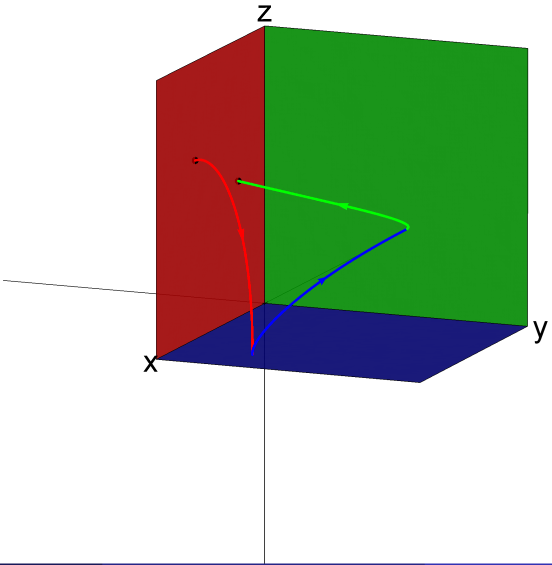
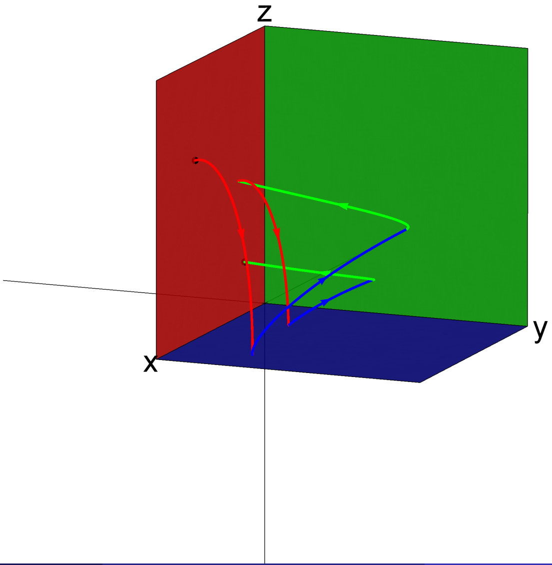
- $\det(M) = -0.293792$.
- $\mathrm{tr}(M) = 0$.

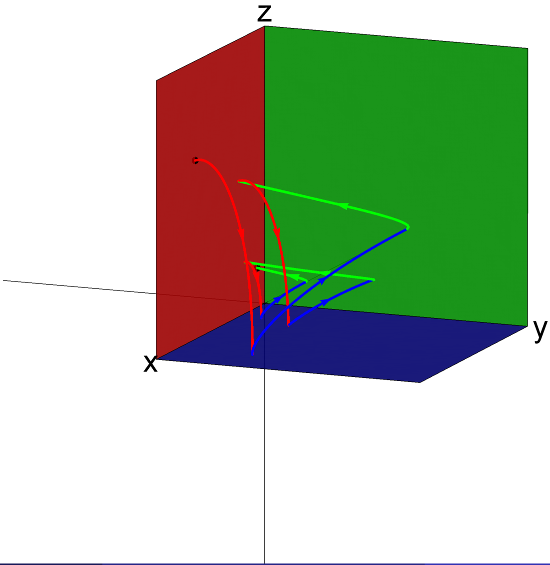
Numerical example



Numerical example



Numerical example



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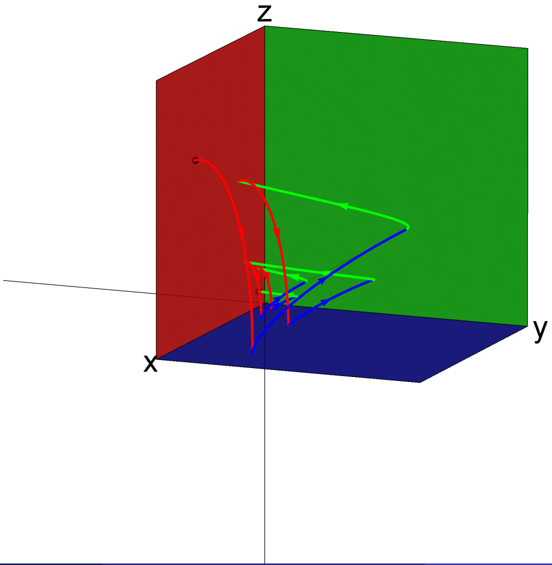

Outline of this section

Conclusions

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Conclusions

- Explicit condition from the data problem.
- Another switching law can be defined by reversing the order of the subsystems. Then, there are two possibilities to check it out.
- In fact, there are not only two switching laws in this method but an infinity number of them can be defined by "switching" the previous two ones... (studying the underlying discrete-time switched linear system)
- An invariant subspace has been provided for this class of switched systems. So we can think about a generalization of further order.

Stabilization for a class of three order switched linear systems

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15 y 16 de octubre 2021