EXACT SOLUTIONS TO OPTIMIZATION PROBLEMS THROUGH SUPPORTING VECTORS ANALYSIS

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Miniworkshop. Órbitas en Análisis Matemático

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SUPPORTING VECTORS

Supporting vectors appear implicitly in the literature of Operator Theory and Banach Space Geometry through famous theorems such as Hahn-Banach, James, Lindenstrauss, Bishop-Phelps-Bollobás, etc.

DEFINITION (SUPPORTING VECTOR)

Let $T : X \to Y$ be a continuous linear operator between normed spaces X, Y. The set of supporting vectors of T is defined as

$$\text{suppv}(T) := \{x \in S_X : \|T(x)\| = \|T\|\}.$$

DEFINITION (EXPOSED FACES)

If $x^* \in X^* \neq 0$, then $\text{suppv}_1(x^*) := \{x \in S_X : x^*(x) = \|x^*\|\}$ are called the exposed faces of B_X.

Topology of supporting vectors

Notice that

$$\mathrm{suppv}(x^*) = \bigcup_{\lambda \in S_{\mathbb{K}}} \lambda \mathrm{suppv}_1(x^*).$$

Remark

If $\mathbb{K} = \mathbb{R}$ and $x^* \neq 0$, then $\{\mathrm{suppv}_1(x^*), -\mathrm{suppv}_1(x^*)\}$ are the only two connected components of $\mathrm{suppv}(x^*)$, hence they are the only two convex components.

Theorem

If $\mathbb{K} = \mathbb{C}$ and $x^* \neq 0$, then $\mathrm{suppv}(x^*)$ is path-connected and the convex components of $\mathrm{suppv}(x^*)$ are $\{\lambda \mathrm{suppv}_1(x^*) : \lambda \in S_{\mathbb{C}}\}$.

Geometry of supporting vectors

Theorem

Let X and Y be normed spaces and let $T : X \longrightarrow Y$ be a non-zero continuous linear operator. Then:

1. $\operatorname{suppv}(T) = \bigcup_{y^* \in \operatorname{suppv}(T^*)} \operatorname{suppv}_1(y^* \circ T)$.
2. If C is a convex component of $\operatorname{suppv}(T)$, then $C = \operatorname{suppv}_1(y^* \circ T)$ for some $y^* \in \operatorname{suppv}(T^*)$.
3. If Y is smooth, then every non-empty $\operatorname{suppv}_1(y^* \circ T)$ with $y^* \in \operatorname{suppv}(T^*)$ is a convex component of $\operatorname{suppv}(T)$.

Theorem

Let X be a normed space. The following are equivalent:

1. The exposed faces of B_X are pairwise disjoint.
2. Every non-empty $\operatorname{suppv}_1(y^* \circ T)$ with $y^* \in \operatorname{suppv}(T^*)$ is a convex component of $\operatorname{suppv}(T)$ for every normed space Y and every non-zero continuous linear operator $T : X \longrightarrow Y$.

Applications of supporting vectors

Theorem

Let X be a Banach space and let $P : X \to X$ be a projection. The following conditions are equivalent:

1. P is an M-projection, that is, $\|x\| = \max\{\|P(x)\|, \|(I-P)(x)\|\}$ for all $x \in X$.
2. P is $(1,1)$, that is, $\|P\| = \|I-P\| = 1$, and $S_X = \mathrm{suppv}(P) \cup \mathrm{suppv}(I-P)$.

Applications of supporting vectors

Definition (Supporting sequence)

Let X and Y be normed spaces and $T : X \longrightarrow Y$ a continuous linear operator. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq S_X$ is called

- a supporting sequence of T provided that $\|T(x_n)\| \longrightarrow \|T\|$ as $n \longrightarrow \infty$,
- and a null sequence for T provided that $\|T(x_n)\| \longrightarrow 0$ as $n \longrightarrow \infty$.

Applications of supporting vectors

Theorem

Let X be a Banach space and $P : X \to X$ a projection. Then:

1. If there exists a supporting sequence of P which is null for $I - P$, then $\|P\| = 1$.
2. If $\operatorname{suppv}(P) \cap \ker(I - P) \neq \varnothing$, then $\|P\| = 1$.
3. If X is uniformly convex and $\|P\| = 1$, then every supporting sequence of P is null for $I - P$.
4. If X is strictly convex and $\|P\| = 1$, then $\operatorname{suppv}(P) \subseteq \ker(I - P)$.

Corollary

Let X be a strictly convex Banach space and $P : X \to X$ a projection. The following conditions are equivalent:

1. $\|P\| = 1$.
2. $\varnothing \neq \operatorname{suppv}(P) \subseteq \ker(I - P)$.
3. $\operatorname{suppv}(P) \cap \ker(I - P) \neq \varnothing$.

Supporting vectors in Hilbert spaces

Theorem

Consider H, K Hilbert spaces, and $T \in \mathcal{B}(H,K)$. Then:

1. $\|T\|^2 = \|T' \circ T\|$.
2. $\operatorname{suppv}(T) \subseteq \operatorname{suppv}(T' \circ T)$.
3. $\operatorname{suppv}(T) \neq \varnothing$ if and only if $\|T' \circ T\| \in \sigma_p(T' \circ T)$.

In this situation, $\|T\| = \sqrt{\lambda_{\max}(T' \circ T)}$ and $\operatorname{suppv}(T) = V(\lambda_{\max}(T' \circ T)) \cap S_H$.

Supporting vectors in ℓ_1−norm

Recall that $\operatorname{supp}(x) := \{n \in \mathbb{N} : x(n) \neq 0\}$.

Theorem

Let $T : \ell_1 \to Y$ be a nonzero continuous linear operator between ℓ_1 and a normed space Y. For every $x \in \ell_1$,

$$T(x) = \sum_{n=1}^{\infty} x(n) T(e_n) \quad \textit{and} \quad \|T(x)\| \leq \sum_{n=1}^{\infty} |x(n)| \|T(e_n)\|.$$

Also, $\|T\| = \sup\{\|T(e_n)\| : n \in \mathbb{N}\}$. As a consequence, $\operatorname{suppv}(T) \neq \varnothing$ *if and only if $N \neq \varnothing$, where $N := \{n \in \mathbb{N} : \|T\| = \|T(e_n)\|\}$. In this situation,*

$$\operatorname{suppv}(T) = \Big\{ y \in S_{\ell_1} : \operatorname{supp}(y) \subseteq N \textit{ and } \left\| \sum_{n \in N} y(n) T(e_n) \right\| = \sum_{n \in N} |y(n)| \|T(e_n)\| \Big\}.$$

Supporting vectors in ℓ_∞−norm

Theorem

Let $T : c_0 \to c_0$ be a nonzero continuous linear operator. For every $x \in c_0$,

$$T(x) = \sum_{i=1}^{\infty} \left(\sum_{n=1}^{\infty} x(n) T(e_n)(i) \right) e_i \quad \text{and} \quad \|T(x)\|_\infty = \sup_{i \in \mathbb{N}} \left| \sum_{n=1}^{\infty} x(n) T(e_n)(i) \right| .$$

Also, $\|T\|_\infty = \sup_{i \in \mathbb{N}} \sum_{n=1}^{\infty} |T(e_n)(i)|$. *As a consequence,* $\operatorname{suppv}(T) \neq \varnothing$ *if and only if* $I_1 \neq \varnothing$*, where*

$$I_1 := \left\{ i_1 \in \mathbb{N} : \sum_{n=1}^{\infty} |T(e_n)(i_1)| = \sup_{i \in \mathbb{N}} \sum_{n=1}^{\infty} |T(e_n)(i)| \text{ and } N_{i_1} \text{ is finite} \right\}$$

with $N_{i_1} := \{ n \in \mathbb{N} : T(e_n)(i_1) \neq 0 \}$. *In this situation,*

$$\operatorname{suppv}(T) = \left\{ \lambda z \in S_{c_0} : |\lambda| = 1 \text{ and } \exists i_1 \in I_1 \ \forall n \in N_{i_1} \ z(n) = \frac{|T(e_n)(i_1)|}{T(e_n)(i_1)} \right\} .$$

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Multiobjective optimization problems

Definition (Multi-objective optimization problem)

Let X be a non-empty set. Let $f_i, g_j : X \longrightarrow \mathbb{R}$, $i = 1, \ldots, p$, $j = 1, \ldots, q$, be functions and let $\mathcal{R}$ be a non-empty subset of X. The problem

$$\begin{cases} \max f_i(x) & i = 1, \ldots, p, \\ \min g_j(x) & j = 1, \ldots, q, \\ x \in \mathcal{R}, \end{cases} \tag{1}$$

is called a *multi-objective optimization problem (MOP).*

- The functions $f_i, g_j : X \longrightarrow \mathbb{R}$, $i = 1, \ldots, p$, $j = 1, \ldots, q$, are called *objective functions*.
- The set $\mathcal{R}$ is called *feasible region*, *region of constrains/restrictions*, or *set of feasible solutions*, and it is often denoted as fsol(1).

Optimal solutions and Pareto solutions

Definition (Optimal solution)

The set of *optimal solutions* of (1) is defined as $\operatorname{osol}(1) := \{x \in \mathcal{R} : \forall i = 1, \ldots, p\ \forall j = 1, \ldots, q\ \forall y \in \mathcal{R}, f_i(x) \geq f_i(y) \text{ and } g_j(x) \leq g_j(y)\}$.

- Due to the often lack of optimal solutions, Pareto solutions are introduced:

Definition (Pareto optimal solution)

The set of *Pareto optimal solutions* of (1) is defined as $\operatorname{psol}(1) := \{x \in \mathcal{R}$: If $y \in \mathcal{R}$ satisfies that there exists $i \in \{1, \ldots, p\}$ with $f_i(y) > f_i(x)$ or exists $j \in \{1, \ldots, q\}$ with $g_j(y) < g_j(x)$, then there exists $i' \in \{1, \ldots, p\}$ with $f_{i'}(y) < f_{i'}(x)$ or exists $j' \in \{1, \ldots, q\}$ with $g_{j'}(x) < g_{j'}(y)\}$.

Splitting the MOP into SOPs

Remark

Note that
$\mathrm{osol}(1) = \mathrm{osol}(P_1) \cap \cdots \cap \mathrm{osol}(P_p) \cap \mathrm{osol}(Q_1) \cap \cdots \cap \mathrm{osol}(Q_q)$,
where

$$P_i := \begin{cases} \max f_i(x), \\ x \in \mathcal{R}, \end{cases} \quad \text{and} \quad Q_j := \begin{cases} \min g_j(x), \\ x \in \mathcal{R}, \end{cases}$$

are single-objective optimization problems (SOPs).

Sorting feasible solutions from less optimal to more optimal

Remark

Consider in $\mathcal{R}$ the equivalence relation given by

$$\mathcal{S} := \left\{(x,y) \in \mathcal{R}^2 : \forall i = 1,\ldots,p,\ \ f_i(x) = f_i(y) \right. \\ \left. \text{and } \forall j = 1,\ldots,q,\ \ g_j(x) = g_j(y)\right\}. \tag{2}$$

Consider in the quotient set of $\mathcal{R}$ by $\mathcal{S}$, $\mathcal{R}/\mathcal{S}$, the order relation given by

$$[x]_{\mathcal{S}} \leq [y]_{\mathcal{S}} \Leftrightarrow \forall i = 1,\ldots,p\ \ f_i(x) \leq f_i(y) \\ \text{and } \forall j = 1,\ldots,q\ \ g_j(y) \leq g_j(x). \tag{3}$$

Topological expression of optimal and Pareto solutions

Theorem

Given a multiobjective optimization problem (1),

$$\mathrm{psol}(1) = \{x \in \mathcal{R} : [x]_{\mathcal{S}} \text{ is a maximal element of } {}^{\mathcal{R}}/_{\mathcal{S}} \text{ endowed with } \leq\}$$

and

$$\mathrm{osol}(1) = \{x \in \mathcal{R} : [x]_{\mathcal{S}} \text{ is the maximum of } {}^{\mathcal{R}}/_{\mathcal{S}} \text{ endowed with } \leq\}.$$

As a consequence, $\mathrm{osol}(1) \subseteq \mathrm{psol}(1)$ and if $\mathrm{osol}(1) \neq \varnothing$, then $\mathrm{osol}(1) = \mathrm{psol}(1)$. Even more, if there exists $i_1 \in \{1, \ldots, p\}$ or $j_1 \in \{1, \ldots, q\}$ such that $\mathrm{osol}(P_{i_1})$ or $\mathrm{psol}(Q_{j_1})$ is a singleton, respectively, then $\mathrm{osol}(P_{i_1}) \subseteq \mathrm{psol}(1)$ or $\mathrm{osol}(Q_{j_1}) \subseteq \mathrm{psol}(1)$, respectively.

Existence of Pareto solutions

Theorem

Given a multiobjective problem (1), set $i_0 \in \{1, \ldots, p\}, j_0 \in \{1, \ldots, q\}$. Then:

1. If there exists $x_{i_0} \in \mathcal{R}$ such that $\left[x_{i_0}\right]_{\mathcal{S}}$ is a maximal element of $\left\{[x]_{\mathcal{S}} : x \in \arg\max_{\mathcal{R}} f_{i_0}\right\}$, then $\left[x_{i_0}\right]_{\mathcal{S}}$ is a maximal element of ${}^{\mathcal{R}}/_{\mathcal{S}}$. Hence, $x_{i_0} \in \operatorname{psol}(1)$.
2. If there exists $x_{j_0} \in \mathcal{R}$ such that $\left[x_{j_0}\right]_{\mathcal{S}}$ is a maximal element of $\left\{[x]_{\mathcal{S}} : x \in \arg\min_{\mathcal{R}} g_{j_0}\right\}$, then $\left[x_{j_0}\right]_{\mathcal{S}}$ is a maximal element of ${}^{\mathcal{R}}/_{\mathcal{S}}$. Hence, $x_{j_0} \in \operatorname{psol}(1)$.

Theorem

Given a multiobjective problem (1)*, if X is a topological space, $\mathcal{R}$ is a compact Hausdorff subset of X and all the objective functions are continuous, then* $\operatorname{psol}(1) \neq \varnothing$.

Reformulations of the original multiobjective problem I

Theorem

Consider the multiobjective problem (1). Suppose that $F : A \subseteq \mathbb{R}^p \to \mathbb{R}$ and $G : B \subseteq \mathbb{R}^q \to \mathbb{R}$ are strictly increasing where $\{(f_1(x), \ldots, f_p(x)) : x \in \mathcal{R}\} \subseteq A$ and $\{(g_1(x), \ldots, g_q(x)) : x \in \mathcal{R}\} \subseteq B$. Consider also the MOP

$$\begin{cases} \max F\left(f_1(x), \ldots, f_p(x)\right), \\ \min G\left(g_1(x), \ldots, g_q(x)\right), \\ x \in \mathcal{R}. \end{cases} \tag{4}$$

Then:

1. $\mathrm{psol}(4) \subseteq \mathrm{psol}(1)$.
2. If $\mathrm{osol}(1) \neq \emptyset$, then $\mathrm{osol}(1) = \mathrm{osol}(4)$.

Reformulations of the original multiobjective problem II

Theorem

Consider the multiobjective problem (1). *Suppose that* $F : A \subseteq \mathbb{R}^p \to \mathbb{R}$ *and* $G : B \subseteq \mathbb{R}^q \to \mathbb{R}$ *are strictly increasing where* $\{(f_1(x), \ldots, f_p(x)) : x \in \mathcal{R}\} \subseteq A$ *and* $\{(g_1(x), \ldots, g_q(x)) : x \in \mathcal{R}\} \subseteq B$. *Suppose also that* $F(f_1(x), \ldots, f_p(x)) > 0$ *and* $G(g_1(x), \ldots, g_q(x)) > 0$ *for all* $x \in \mathcal{R}$. *Consider the SOP*

$$\begin{cases} \max \dfrac{F(f_1(x), \ldots, f_p(x))}{G(g_1(x), \ldots, g_q(x))}, \\ x \in \mathcal{R}. \end{cases} \qquad (5)$$

Then:

1. osol(5) $\subseteq$ psol(1).
2. *If* osol(1) $\neq \emptyset$, *then* osol(1) = osol(5).

Reformulations of the original multiobjective problem III

Recall that a family of real-valued functions $\{h_k : k \in K\}$ defined on a set A is not simultaneously zero if for every $a \in A$ there exists $k \in K$ such that $h_k(a) \neq 0$. In other words, $\bigcap_{k \in K} h_k^{-1}(\{0\}) = \emptyset$.

Corollary

Consider the multiobjective problem (1). *Suppose that* $f_i(x), g_j(x) \geq 0$ *for all* $x \in \mathcal{R}$*, all* $i \in \{1, \ldots, p\}$ *and all* $j \in \{1, \ldots, q\}$*, and that the families* $\{f_1, \ldots, f_p\}$ *and* $\{g_1, \ldots, g_q\}$ *are not simultaneously zero in* $\mathcal{R}$*. Consider the SOP*

$$\begin{cases} \max \dfrac{f_1(x)^2 + \cdots + f_p(x)^2}{g_1(x)^2 + \cdots + g_q(x)^2}, \\ x \in \mathcal{R}. \end{cases} \tag{6}$$

Then:

1. $\operatorname{osol}(6) \subseteq \operatorname{psol}(1)$.
2. *If* $\operatorname{osol}(1) \neq \emptyset$*, then* $\operatorname{osol}(1) = \operatorname{osol}(6)$.

Operator Multiobjective Optimization Problems

Definition (Operator MOP)

An *operator multi-objective problem (OMOP)* is a special type of MOP given by

$$\left\{ \begin{array}{l} \max \|T_i(x)\| \quad i = 1, \ldots, p, \\ \min \|S_j(x)\| \quad j = 1, \ldots, q, \\ x \in \mathcal{R}, \end{array} \right. \tag{7}$$

where $T_i, S_j : X \longrightarrow Y$ are continuous linear operators, $i = 1, \ldots, p$, $j = 1, \ldots, q$, between normed spaces X, Y, and $\mathcal{R}$ is a non-empty subset of X.

Simplification theorems on the OMOP

Remark

The functions

$$\begin{array}{rrcl} F: & [0,\infty)^p & \longrightarrow & \mathbb{R} \\ & (x_1,\ldots,x_p) & \mapsto & F(x_1,\ldots,x_p) := \|(x_1,\ldots,x_p)\|_r = \left(\sum_{i=1}^p x_i^r\right)^{\frac{1}{r}} \end{array}$$

$$\begin{array}{rrcl} G: & [0,\infty)^q & \longrightarrow & \mathbb{R} \\ & (x_1,\ldots,x_q) & \mapsto & G(x_1,\ldots,x_q) := \|(x_1,\ldots,x_q)\|_r = \left(\sum_{i=1}^q x_i^r\right)^{\frac{1}{r}} \end{array}$$

are strictly increasing, where $1 \leq r < \infty$.

Simplification theorems on the OMOP

Corollary

Consider the OMOP (7). According to Theorem 18, psol(8) $\subseteq$ psol(7), and if osol(7) $\neq \emptyset$, then osol(7) = osol(8), where

$$\left\{\begin{array}{l} \max \sqrt[r]{\|T_1(x)\|^r + \cdots + \|T_p(x)\|^r} \\ \min \sqrt[r]{\|S_1(x)\|^r + \cdots + \|S_q(x)\|^r} \\ x \in \mathcal{R}. \end{array}\right. = \left\{\begin{array}{l} \max \|T(x)\|_r \\ \min \|S(x)\|_r \\ x \in \mathcal{R} \end{array}\right. \quad (8)$$

Corollary

Consider the OMOP (7). If $\mathcal{R} \subseteq X \setminus \left(\bigcap_{i=1}^{p} \ker(T_i) \cup \bigcap_{j=1}^{q} \ker(S_j)\right)$. According to Theorem 19, osol(9) $\subseteq$ psol(7), and if osol(7) $\neq \emptyset$, then osol(7) = osol(9), where

$$\left\{\begin{array}{l} \max \dfrac{\sqrt{\|T_1(x)\|^r + \cdots + \|T_p(x)\|^r}}{\sqrt{\|S_1(x)\|^r + \cdots + \|S_q(x)\|^r}} \\ x \in \mathcal{R} \end{array}\right. = \left\{\begin{array}{l} \max \dfrac{\|T(x)\|_r}{\|S(x)\|_r} \\ x \in \mathcal{R} \end{array}\right. \quad (9)$$

Simplification theorems on the OMOP

where

$$\begin{array}{rccl} T: & X & \longrightarrow & \ell_r^p(Y) := Y \oplus_r \overset{p}{\cdots} \oplus_r Y \\ & x & \mapsto & T(x) := \left(T_1(x), \ldots, T_p(x)\right) \end{array}$$

and

$$\begin{array}{rccl} S: & X & \longrightarrow & \ell_r^q(Y) := Y \oplus_r \overset{q}{\cdots} \oplus_r Y \\ & x & \mapsto & S(x) := \left(S_1(x), \ldots, S_q(x)\right). \end{array}$$

Simplification theorems on the OMOP

Consider the MOP

$$\begin{cases} \max \|T(x)\|, \\ \min \|S(x)\|, \\ x \in X. \end{cases} \tag{10}$$

and the SOPs

$$A_1 := \begin{cases} \max \|T(x)\|, \\ \|S(x)\| \leq 1, \end{cases} \quad B_1 := \begin{cases} \min \|S(x)\|, \\ \|T(x)\| \geq 1, \end{cases}$$

$$C_1 := \begin{cases} \min \frac{\|S(x)\|}{\|T(x)\|}, \\ \|T(x)\| \neq 0, \end{cases} \quad D_1 := \begin{cases} \max \frac{\|T(x)\|}{\|S(x)\|}, \\ \|S(x)\| \neq 0. \end{cases}$$

Simplification theorems on the OMOP

Theorem

Let X, Y be normed spaces and $T, S : X \to Y$ continuous linear operators. Then:

1. If $\ker(S) \setminus \ker(T) \neq \varnothing$, then $\mathrm{psol}(10) = \mathrm{osol}(A_1) = \varnothing$.
2. If $\ker(S) \subseteq \ker(T) \subsetneq X$, then $\mathrm{osol}(A_1) \subseteq \{x \in X : \|S(x)\| = 1\}$.
3. If $\ker(S) \subseteq \ker(T)$, then $\ker(S) \subseteq \mathrm{psol}(10)$ and $\mathrm{psol}(10) = \mathbb{R}\mathrm{psol}(10)$.
4. $\mathrm{psol}(10) \setminus \ker(S) \subseteq \mathbb{R}^+\mathrm{osol}(A_1)$.
5. If $\ker(S) \subseteq \ker(T) \subsetneq X$, then $\mathrm{osol}(A_1) \subseteq \mathrm{psol}(10)$.

Simplification theorems on the OMOP

Theorem

Let X, Y be normed spaces and $T, S : X \longrightarrow Y$ nonzero continuous linear operators. Then:

1. $\mathrm{osol}(D_1) = \bigcup_{t>0} t\,\mathrm{osol}(A_1)$.
2. $\mathrm{osol}(C_1) = \bigcup_{t>0} t\,\mathrm{osol}(B_1)$.
3. If $\mathrm{osol}(A_1) \neq \emptyset$, then $\ker(S) \subseteq \ker(T)$.
4. If X is finite dimensional, then $\mathrm{osol}(A_1) \neq \emptyset$ if and only if $\ker(S) \subseteq \ker(T)$.
5. If $\ker(S) \subseteq \ker(T)$, then $\mathrm{osol}(C_1) = \mathrm{osol}(D_1)$.
6. If $\ker(S) \setminus \ker(T) \neq \emptyset$, then $\mathrm{osol}\,(B_1) = \ker(S) \setminus U_{T(X)}$ and $\mathrm{osol}\,(C_1) = \ker(S) \setminus \ker(T)$.

Simplification theorems on the OMOP

Corollary

If $T \neq 0$, then $\mathbb{R}\mathrm{suppv}(T) = \mathrm{psol}(11)$, where

$$\left\{\begin{array}{l} \max \|T(x)\|, \\ \min \|x\|, \\ x \in X. \end{array}\right. \tag{11}$$

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EXAMPLE

$$\left\{\begin{array}{l}\max \|E_x\psi\|_2\\ \max \|E_y\psi\|_2\\ \max \|E_z\psi\|_2\\ \min \psi^T R\psi\end{array}\right. = \left\{\begin{array}{l}\max \|E_x\psi\|_2^2\\ \max \|E_y\psi\|_2^2\\ \max \|E_z\psi\|_2^2\\ \min \|C\psi\|_2^2 \quad (R = C^T C)\end{array}\right. \overset{\textit{Theorem } 18}{\Longrightarrow}$$

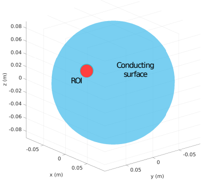
$$\left\{\begin{array}{l}\max \|E_x\psi\|_2^2 + \|E_y\psi\|_2^2 + \|E_z\psi\|_2^2\\ \min \|C\psi\|_2^2\end{array}\right. = \left\{\begin{array}{l}\max \|E\psi\|_2\\ \min \|C\psi\|_2\end{array}\right. \quad E = \begin{pmatrix} E_x\\ E_y\\ E_z\end{pmatrix}$$

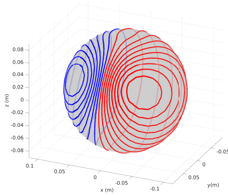
$$\overset{\textit{Theorem } 24}{\Longrightarrow} \left\{\begin{array}{l}\max \|E\psi\|_2\\ \|C\psi\|_2 \leq 1\end{array}\right. = \left\{\begin{array}{l}\max \|EC^{-1}\phi\|_2^2\\ \|\phi\|_2 \leq 1\end{array}\right. \overset{\textit{Theorem } 8}{\Longrightarrow}$$

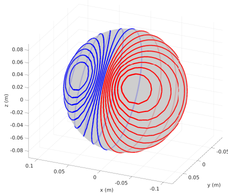
SOLUTION

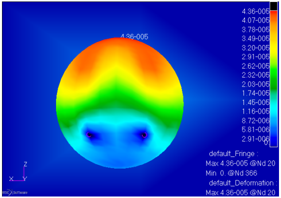
$\psi = C^{-1}\phi$ where ϕ is a unit eigenvector of $\lambda_{\max}\left(\left(EC^{-1}\right)^T \left(EC^{-1}\right)\right)$

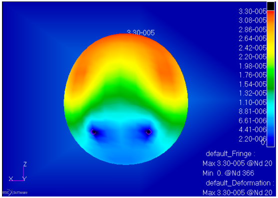
Optimal TMS coils





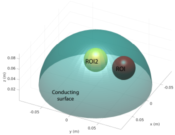


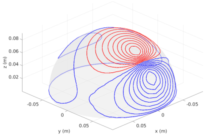


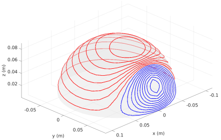


	L (μH)	R ($m\Omega$)	$D_{1/2}$ (cm)	$S_{1/2}$ (cm^2)	$\lvert \mathbf{f}_{max} \rvert$ (N·m^{-3})	σ_{max} (MPa)
CoilS0	18.1	134	2.01	20.3	4.30×10^8	1.09
CoilS	19.0	152	2.15	22.2	2.79×10^8	0.94

Optimal TMS coils

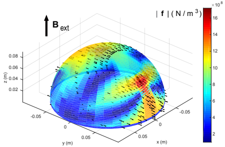






	f_{max} (10^9 N·m^{-3}) with $\mathbf{B}_{ext} = 3$ T $\hat{\mathbf{i}}$	f_{max} (10^9 N·m^{-3}) with $\mathbf{B}_{ext} = 3$ T $\hat{\mathbf{j}}$	f_{max} (10^9 N·m^{-3}) with $\mathbf{B}_{ext} = 3$ T $\hat{\mathbf{k}}$
CoilH0	1.33	2.24	1.72
CoilHz	1.21	1.51	1.52

	L (μH)	R ($m\Omega$)	$D_{1/2}$ (cm)	$S_{1/2}$ (cm^2)	$N_{contours}$
CoilH0	9.7	89	1.29	16.4	18
CoilHz	14.6	106	1.94	18.4	18



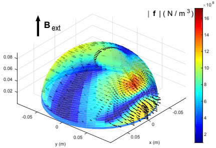


Table of contents

Publications I

- Clemente Cobos-Sánchez, Francisco Javier García-Pacheco, Soledad Moreno-Pulido, and Sol Sáez-Martínez.
  Supporting vectors of continuous linear operators.
  Annals of Functional Analysis, 8(4):520 – 530, 2017.

- Francisco Javier Garcia-Pacheco, Clemente Cobos-Sanchez, Soledad Moreno-Pulido, and Alberto Sanchez-Alzola.
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  Communications in Nonlinear Science and Numerical Simulation, 82:105054, 2020.

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- Alberto Sánchez-Alzola, Francisco Javier García-Pacheco, Enrique Naranjo-Guerra, and Soledad Moreno-Pulido.
  Supporting vectors for the ℓ_1-norm and the ℓ_∞-norm and an application.
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- Alberto Sánchez-Alzola, Soledad Moreno-Pulido, Enrique Naranjo-Guerra, and Francisco Javier García-Pacheco.
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- Francisco Javier García-Pacheco, Clemente Cobos-Sanchez, Almudena Campos-Jiménez, José Antonio Vilchez-Membrilla, Soledad Moreno-Pulido, and Alberto Sánchez-Alzola.
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