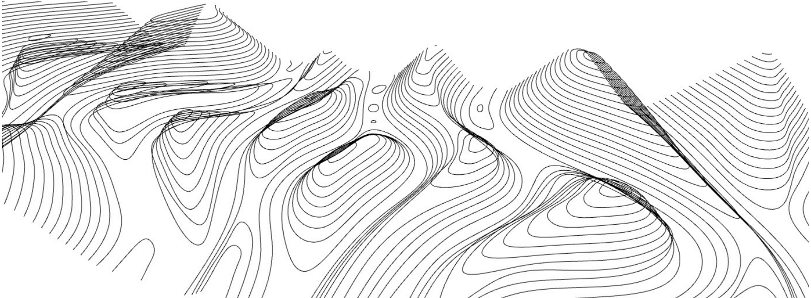
ODEs from a geometric viewpoint

Antonio J. Pan-Collantes

October 15, 2021



Index

ODEs

Equation:

$$u^{m)}(x) = F(x, u(x), u'(x), \ldots, u^{m-1)}(x)) \text{ and}$$

$$\begin{cases} u(0) = c_0, \\ u'(0) = c_1 \\ \ldots \\ u^{m-1)}(0) = c_{m-1} \end{cases}$$

where $F : \mathbb{R}^{m+1} \to \mathbb{R}$ with conditions...

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Solutions:

$$¿u : D \subseteq \mathbb{R} \to \mathbb{R}?$$

Flow of vector fields

Fundamental theorem on flows

Given a vector field X on $\mathbb{R}^n$, then X has a unique maximal flow

$$\phi_t^X : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$$

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Local group of transformations

Vector fields $\Longleftrightarrow$ local groups of transformations.

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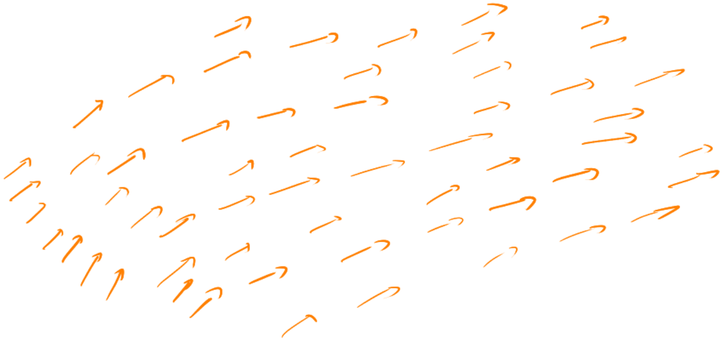
$$\text{Vector fields} \Longleftrightarrow \text{local groups of transformations.}$$

ODE Existence, Uniqueness and Smoothness

$$\begin{cases} \dot{x}_1(t) & = V_1(x_1, \dots, x_n) \\ \quad \cdots & = \cdots \\ \dot{x}_n(t) & = V_n(x_1, \dots, x_n) \end{cases}$$

[Lee, 2013], [Olver, 1986]

Visualization



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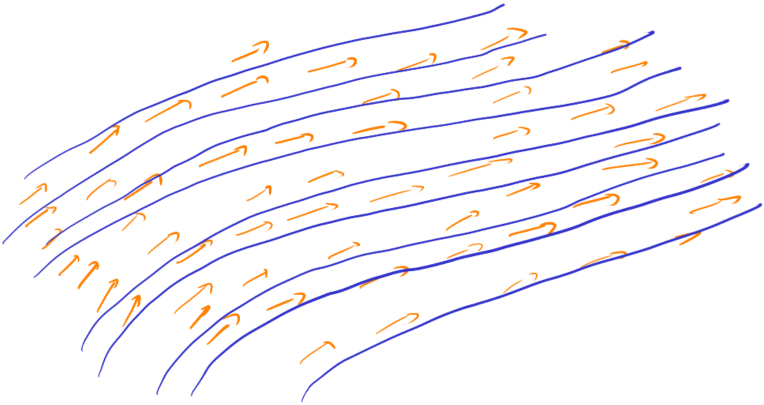

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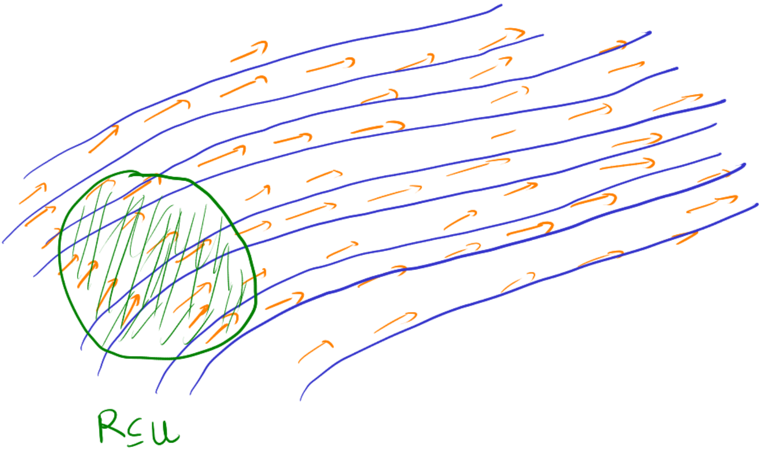

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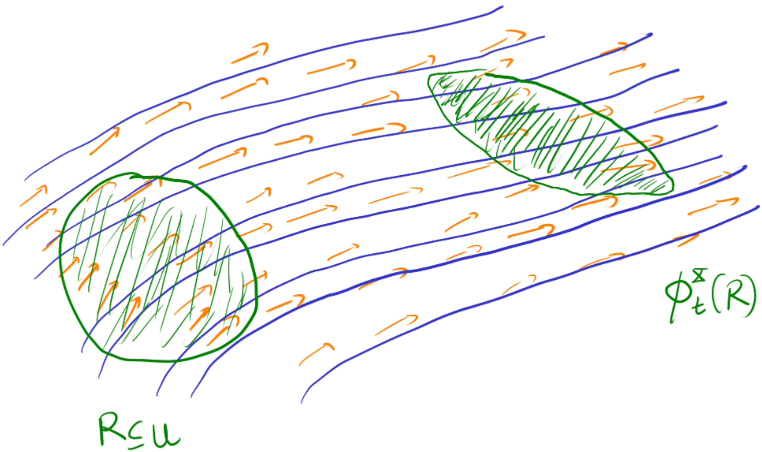

Linkage?

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$$A = \partial x + u_1 \partial u + \ldots + F \partial u_{m-1},$$

with flow ϕ_x^A

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Lie ideas

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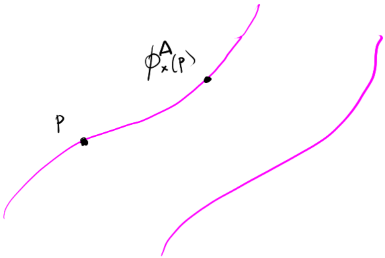

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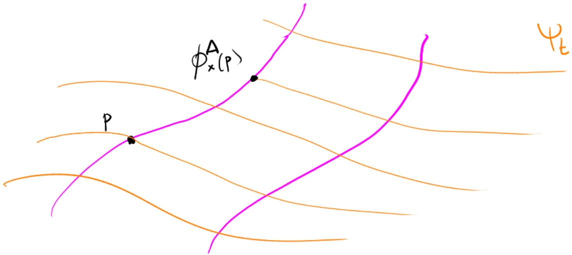

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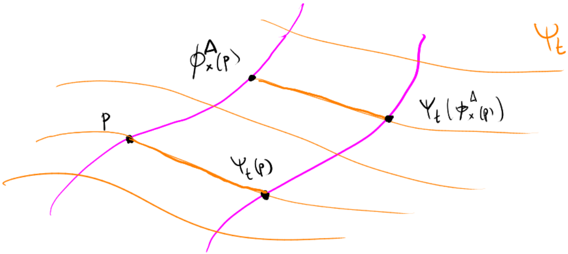

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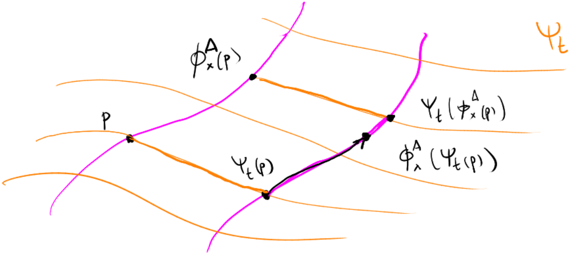

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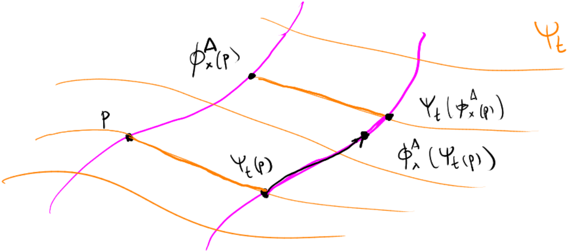

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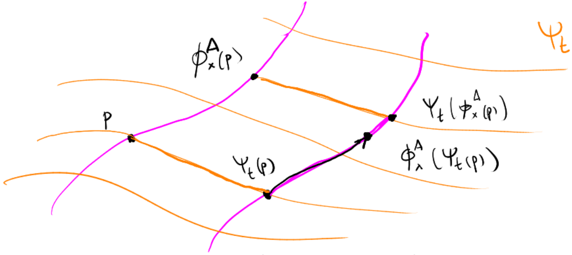

$$\phi_x^A(\Psi_t(p)) = \Psi_t(\phi_x^A(p)) \Rightarrow [X, A] = 0$$

$$\phi_x^A(\Psi_t(p)) \neq \Psi_t(\phi_x^A(p)) \Rightarrow [X, A] = \alpha \cdot A$$

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$$\frac{du}{dx} = F(x, u(x)) \longleftrightarrow -F dx + du = 0$$

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From our point of view

Single line from a vector

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$$\begin{vmatrix} 3 & x \\ 5 & u \end{vmatrix} = -5x + 3u = 0$$

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$$A = \partial x + F\partial u = (1, F)$$

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Family of lines from family of equations

$$\begin{vmatrix} 1 & dx \\ F & du \end{vmatrix} = -Fdx + du = 0$$

Is this 1-form exact?

Exact 1-form

There exists a function $G : \mathbb{R}^2 \to \mathbb{R}$ such that $dG = -Fdx + du$

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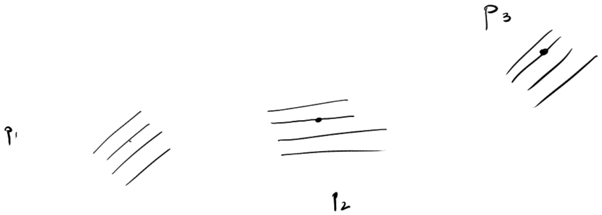

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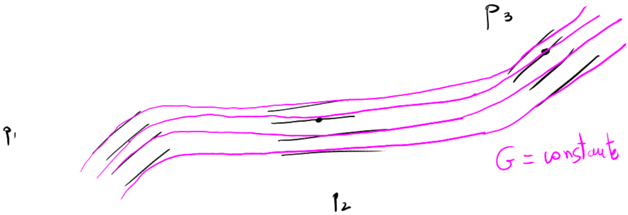

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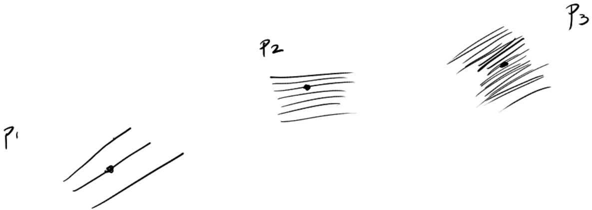

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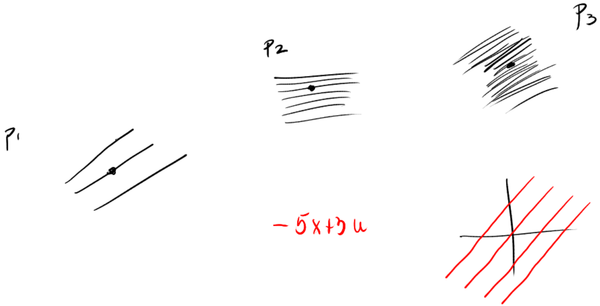

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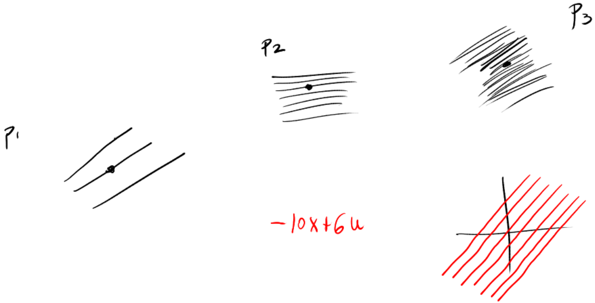

Integrating factor

μ such that there exists G with $dG = \mu(-F dx + du)$

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Key result

If X is a Lie symmetry for A then

$$\frac{1}{\det(A, X)}$$

is an integrating factor for $-Fdx + du$, or in other words,

$$\omega = \frac{\det(A, -)}{\det(A, X)}$$

is exact (locally).

Idea of the proof

det (A, Y)

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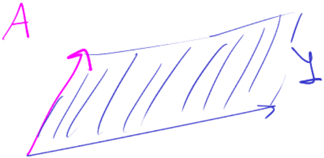
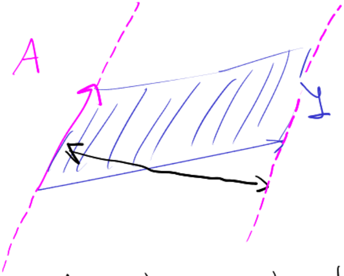


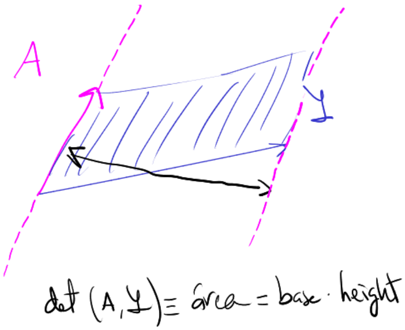
$\det(A, Y) \equiv$ área

Idea of the proof



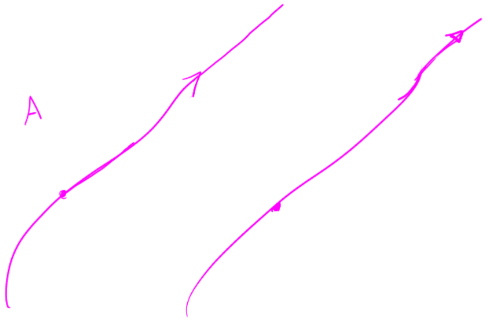
$\det(A, Y) = \text{area} = \text{base} \cdot \text{height}$

Idea of the proof



But base changes when we move along solution curves!

Idea of the proof



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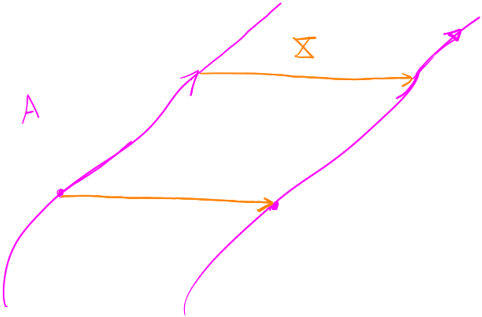

$$[\mathbb{X}, A] = \alpha \cdot A$$

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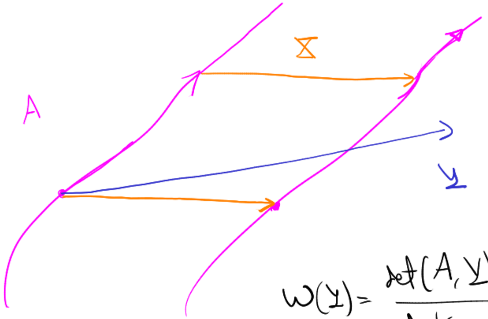

$$[X, A] = \alpha \cdot A$$

$$\omega(Y) = \frac{\det(A, Y)}{\det(A, X)}$$

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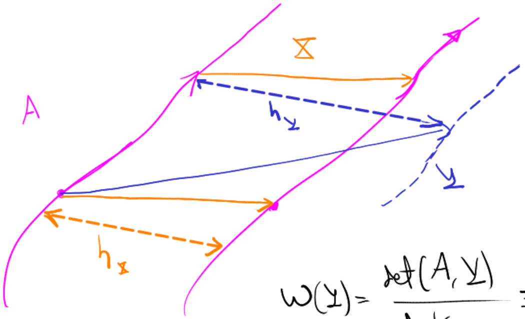

$$[X, A] = \alpha \cdot A$$

$$W(Y) = \frac{\det(A, Y)}{\det(A, X)} = \frac{\text{base} \cdot h_Y}{\text{base} \cdot h_X} = \frac{h_Y}{h_X}$$

Higher order equations

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Solvable structures: [Basarab-Horwath, 1991]

- Symmetries of involutive distributions instead of vector fields

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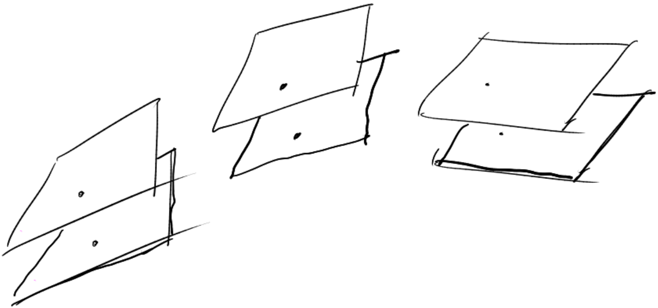
Solvable structures: [Basarab-Horwath, 1991]

- Symmetries of involutive distributions instead of vector fields
- Given A, we aim to an ordered collection $\langle X_1 \ldots, X_k \rangle$ such that X_j is symmetry of the "previous distribution".

Idea



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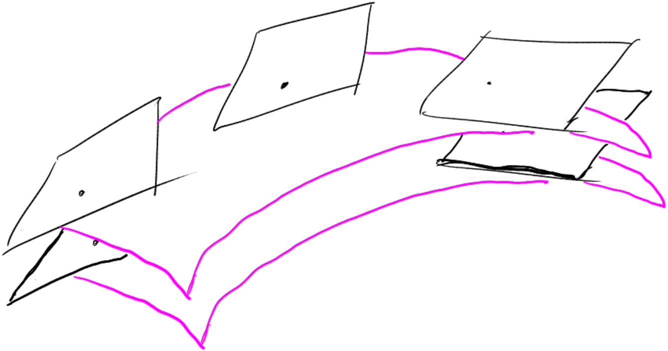

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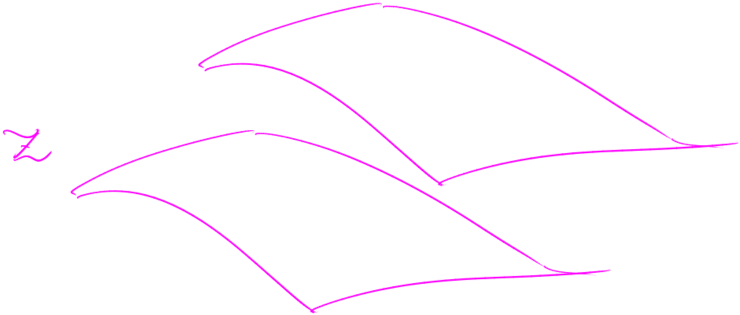

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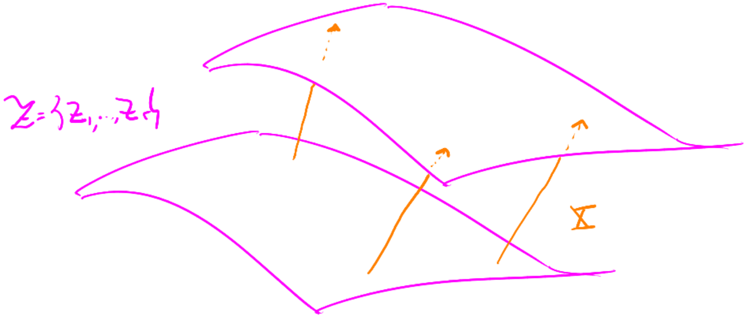
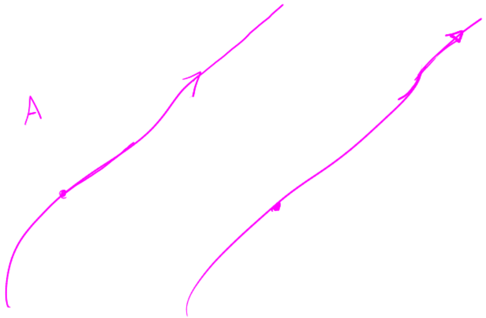


$$[X, Z_i] = \sum \alpha_j Z_j$$

Idea



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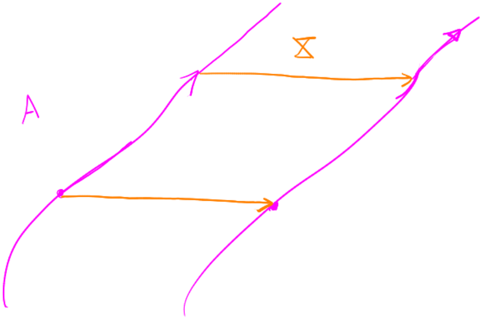

$$[X, A] = \alpha \cdot A$$

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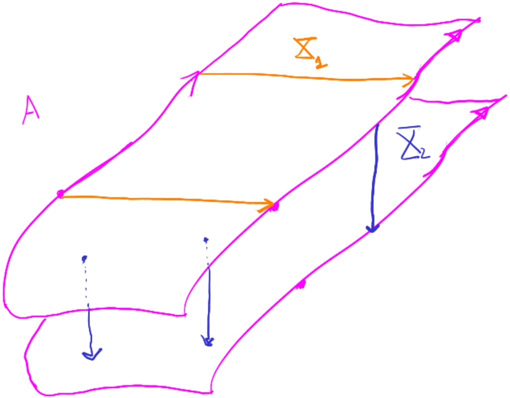

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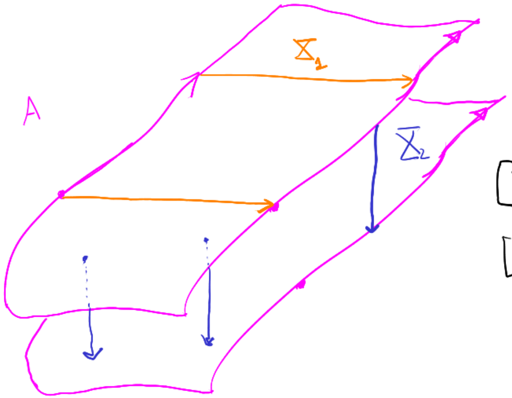

$$[X_1, A] = \alpha \cdot A$$

$$[X_2, A] = \alpha_{21} A + \alpha_{22} X_1$$

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Main result

[Basarab-Horwath, 1991]

Given an ODE by the vector field A, the knowledge of a solvable structure for A is equivalent to the integrability by quadratures of the equation (locally).

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- Integrate and restring to hypersurfaces.

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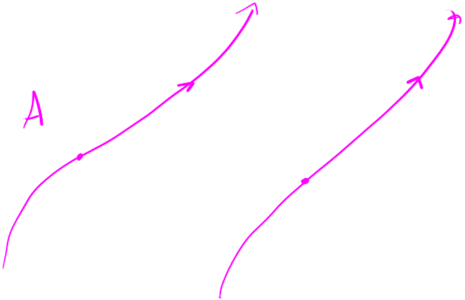
Or, in other words, there exists a (dynamical) symmetry X such that $Y = \frac{1}{f} X$.

Visual idea



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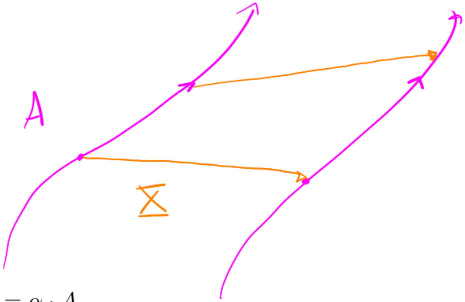

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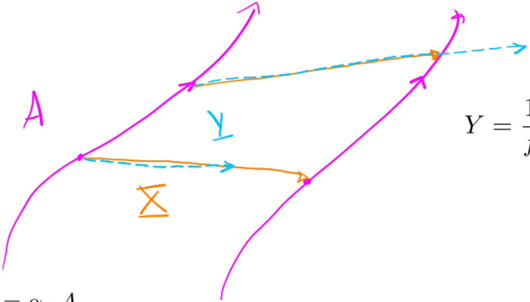
Visual idea

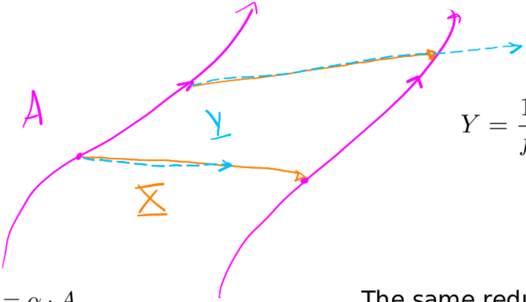


$$Y = \frac{1}{f} \cdot X$$

$$[X, A] = \alpha \cdot A$$

$$[Y, A] = \lambda \cdot Y + \alpha \cdot A$$

Visual idea



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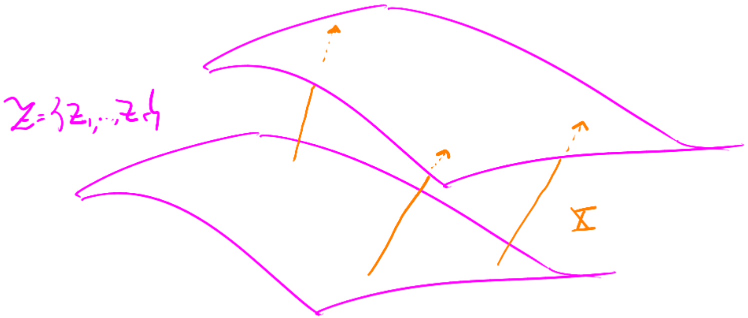
The same reduction procedure that with Lie symmetries

In progress: $\mathcal{C}^\infty$-symmetry of a distribution



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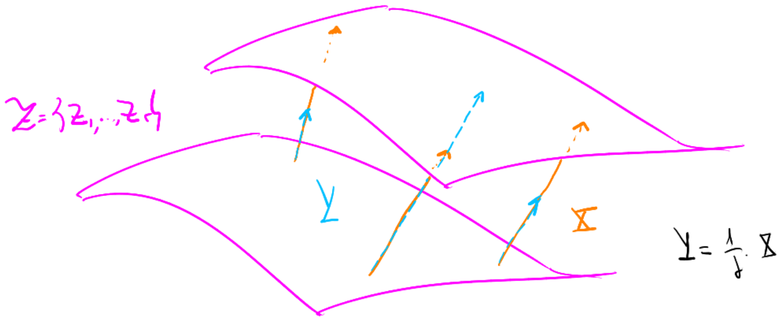

$$[X, Z_i] = \sum \alpha_j Z_j \qquad [Y, Z_i] = \lambda_i Y + \sum \alpha_j Z_j$$

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