2-iso-reflexivity of pointed Lipschitz spaces

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The concept of 2-local isometry

In the last decades considerable work has been done on

General question

Given a class C of transformation (like derivations, automorphisms or isometries),

is C determined by its local actions?

In other words,

1. if $\phi : E \to F$ is a linear map such that, for all $u \in E$, there exists $T_u \in C$ with $\phi(u) = T_u(u)$;

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is $\mathcal{C}$ determined by its local actions?

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2. if $\Delta : E \to F$ is a map such that, for all $u, v \in E$, there exists $T_{u,v} \in \mathcal{C}$ with $\Delta(u) = T_{u,v}(u)$ and $\Delta(v) = T_{u,v}(v)$;

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Definition (2-local isometry)

Let E, F be Banach spaces. A map $\Delta\colon E \to F$ (no linearity nor surjectivity are assumed) is called a 2-local isometry if for every $u, v \in E$, there exists a surjective linear isometry $T_{u,v}\colon E \to F$ such that

$$\Delta(u) = T_{u,v}(u), \qquad \Delta(v) = T_{u,v}(v).$$

Every 2-local isometry Δ preserves the distance between points.

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Is Δ linear and surjective?

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2-local derivations

He proved → If H is an infinite-dimensional separable Hilbert space, then every 2-local automorphism of the C^*-algebra $B(H)$ of all bounded linear operators on H is an automorphism.

Similar assertion holds concerning the 2-local derivations.

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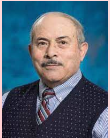

(2007)

Hatori, **Miura**, **Oka** and **Takagi** study 2-local isometries and 2-local automorphisms on uniform algebras (and, in particular, for certain algebras of holomorphic functions).

O. Hatori

T. Miura

(2020)

Hosseini described 2-local isometries on spaces of functions of bounded variation.

Lipschitz and pointed Lipschitz spaces

Lipschitz map

Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is said to be Lipschitz if there exists a constant $\mathrm{C} \geq 0$ such that

$$d_Y(f(x), f(p)) \leq \mathrm{C}\, d_X(x, p) \qquad (x, p \in X).$$

In such case, the number

$$L(f) = \sup\left\{\frac{d_Y(f(x), f(p))}{d_X(x, p)} : x, p \in X,\ x \neq p\right\}$$

is called the Lipschitz constant of f.

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The Lipschitz space $\mathrm{Lip}(X)$

Let X be a metric space. $\mathrm{Lip}(X)$ stands for the set of all bounded Lipschitz functions $f\colon X \to \mathbb{K}$ (where $\mathbb{K}$ denotes the field of real or complex numbers), equipped with either the maximum norm $\max\{\|f\|_\infty, L(f)\}$ or the sum norm $\|f\|_\infty + L(f)$.

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- $\mathrm{Lip}(X)$ is a Banach space.

A pointed metric space is a metric space X with a distinguished element $e_X \in X$ called base point.

The pointed Lipschitz space $\mathrm{Lip}_0(X)$

Let X be a pointed metric space with base point e_X. The pointed Lipschitz space $\mathrm{Lip}_0(X)$ is the Banach space of all Lipschitz functions $f : X \to \mathbb{K}$ for which $f(e_X) = 0$, endowed with the Lipschitz norm $L(f)$.

The isometry group of $\mathrm{Lip}(X)$ is said to be canonical if every surjective linear isometry $T\colon \mathrm{Lip}(X) \to \mathrm{Lip}(X)$ can be expressed as a weighted composition operator of the form

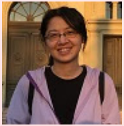
$$T(f) = \lambda \cdot (f \circ \phi) \qquad (f \in \mathrm{Lip}(X)),$$

where λ is an unimodular constant and ϕ is a surjective isometry of X.

(2011)

Jiménez, Villegas ⟶ If X is bounded and separable and the isometry group of $\mathrm{Lip}(X)$ is canonical, then every 2-local isometry of $\mathrm{Lip}(X)$ is a surjective linear isometry.

(2018)
Jiménez, **Li**, **Peralta**, **Wang** and **Wang** studied 2-local isometries between spaces of vector-valued Lipschitz functions.

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   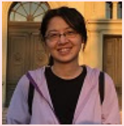

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Recently, **Oi** has been extended the spherical variant of the Kowalski–Słodkowski theorem and she has applied it to prove that 2-local maps in the set of all surjective isometries on several function spaces are surjective isometries.

Surjective linear isometries on Lip_0 spaces

Our problem

Is every 2-local isometry between Lip_0 spaces linear and surjective?

We first obtain a representation of the 2-local isometries between Lip_0 spaces by following:

- The strategy of Győry on $C_0(X)$.
- The technique employed by Győry and Molnár, and Cabello Sánchez to describe the form of diameter-preserving linear bijections of $C(X)$.

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Györy used $\longrightarrow$ The Banach-Stone theorem:

If X is a locally compact Hausdorff space and $T\colon C_0(X) \to C_0(X)$ is a surjective linear isometry, then there exist a homeomorfism $\varphi\colon X \to X$ and a continuous function $\tau\colon X \to \mathbb{K}$ with $|\tau(x)| = 1$ for all $x \in X$ such that

$$T(f) = \tau \cdot (f \circ \varphi) \qquad (f \in C_0(X)).$$

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Mayer-Wolf characterized the surjective linear isometries from $\mathrm{Lip}_0(X, d_X^\alpha)$ onto $\mathrm{Lip}_0(Y, d_Y^\alpha)$ for $\alpha \in]0,1[$.

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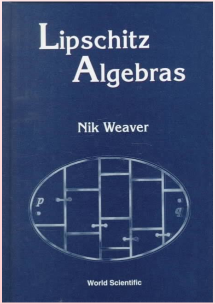
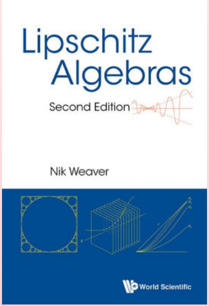
(1999) and (2018)
Weaver extended Mayer-Wolf's result in his books (Theorem 3.56 (2018), Theorem 2.7.3 (1999) and Theorem 3.39 (2018)).





Weaver uses:

Concave and uniformly concave metric space

A metric space X is said to be

- concave if
$$d(x,y) < d(x,z) + d(z,y)$$
for any triple of distinct points $x, y, z \in X$;
- uniformly concave if for every distinct points $x, y \in X$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that
$$d(x,y) \leq d(x,z) + d(z,y) - \delta$$
for all $z \in X$ such that $\min\{d(x,z), d(y,z)\} \geq \varepsilon$.

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Easy example of uniformly concave metric space

The unit circumference $X = S_{(\mathbb{R}^2, \|\cdot\|_2)}$ is uniformly concave.

Indeed, given $x, y \in X$ and $0 < \varepsilon < 2$, we can take $\delta = \left(1 - \sqrt{1 - \varepsilon^2/4}\right) d(x, y)$.

Then $d(x, y) \leq d(x, z) + d(z, y) - \delta$ for all $z \in X$ such that $\min\{d(x, z), d(y, z)\} \geq \varepsilon$.

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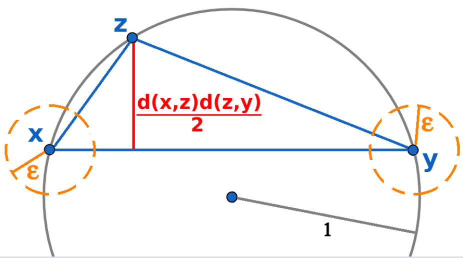

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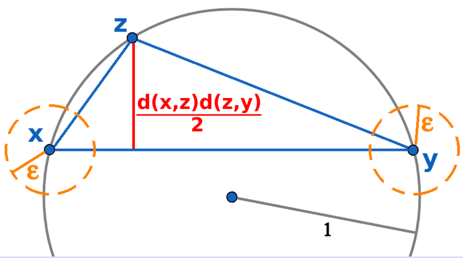

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Weaver (2018) offers:

Examples of uniformly concave metric spaces

1. Any closed subset of $\mathbb{R}^n$ with the inherited Euclidean norm in which no three points are colinear.
2. Any compact subset of a strictly convex Banach space in which no three points are colinear.
3. The unit sphere of any uniformly convex Banach space.
4. Any metric space $(X, \omega \circ d)$, where $\omega\colon (0,\infty) \to (0,\infty)$ is a strictly concave distortion function. In particular, any Hölder metric space (X, d^α) with $\alpha \in]0,1[$.

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Given two metric spaces (X, d_X) and (Y, d_Y) and a number $a > 0$, a map $\phi\colon Y \to X$ is an *a*-dilation if $d_X(\phi(y_1), \phi(y_2)) = a \cdot d_Y(y_1, y_2)$ for all $y_1, y_2 \in Y$.

Theorem (Weaver)

Let X and Y be uniformly concave complete pointed metric spaces. A linear operator $T\colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ is a surjective isometry if and only if there exists a number $\lambda \in S_{\mathbb{K}}$ and a surjective a-dilation $\phi\colon Y \to X$ such that

$$T(f)(y) = \lambda a^{-1} \left(f(\phi(y)) - f(\phi(e_Y))\right)$$

for all $f \in \mathrm{Lip}_0(X)$ and $y \in Y$.

Representation of the 2-local isometries between Lip_0 spaces

Another important tool: peaking functions of $\mathrm{Lip}_0(X)$

Let X be a concave pointed metric space and $x, p \in X$ with $x \neq p$. Consider the functions $g_{(x,p)}, h_{(x,p)} : X \to \mathbb{R}$ defined by

$$g_{(x,p)}(z) = \frac{d(x,p)\, d(z,p)}{d(z,x) + d(z,p)}, \quad h_{(x,p)}(z) = g_{(x,p)}(z) - g_{(x,p)}(e_X)$$

for all $z \in X$. Then $h_{(x,p)}$ belongs to $\mathrm{Lip}_0(X)$, and satisfies that

$$\frac{h_{(x,p)}(x) - h_{(x,p)}(p)}{d(x,p)} = 1, \quad \frac{\left|h_{(x,p)}(z) - h_{(x,p)}(w)\right|}{d(z,w)} < 1$$

for all $z, w \in X$ with $z \neq w$ and $\{z, w\} \neq \{x, p\}$.

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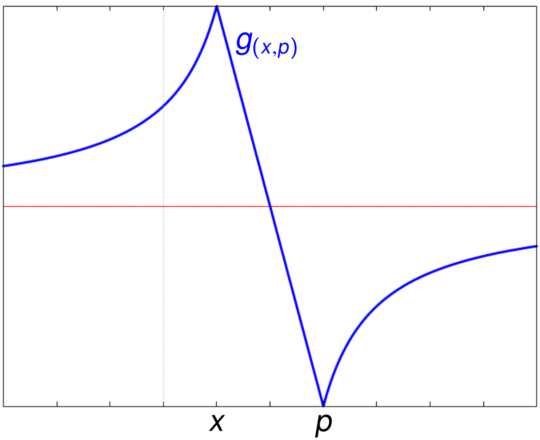

Lemma (more peaking functions)

Let X be a concave pointed metric space, $x, p \in X$ with $x \neq p$ and $0 < \delta < d(x,p)$. Consider the functions $g_1, g_2, g_3 \colon X \to \mathbb{R}$ defined by

$$g_1(z) = \frac{2d(x,p) - \delta}{2d(x,p)} \max\{0, d(x,p) - d(z,x)\} - \frac{\delta}{2d(x,p)} \max\{0, d(x,p) - d(z,p)\},$$

$$g_2(z) = \max\left\{g_1(z), -\frac{1}{2}\max\{0, \delta - d(z,p)\}\right\},$$

$$g_3(z) = \min\left\{g_2(z), \frac{4d(x,p) - 2\delta}{4d(x,p) - \delta}\max\left\{0, d(x,p) - \frac{\delta}{4} - d(z,x)\right\}\right\}.$$

Lemma (more peaking functions (cont.))

Then

1. For each $k \in \{1,2,3\}$, the function g_k is Lipschitz with
$$\frac{g_k(x) - g_k(p)}{d(x,p)} = 1, \qquad \frac{\left|g_k(z) - g_k(w)\right|}{d(z,w)} < 1$$
for all $z, w \in X$ with $z \neq w$ and $\{z,w\} \neq \{x,p\}$.
2. $g_3(z) = 0$ if $d(z,x) \geq d(x,p) - \delta/4$ and $d(z,p) \geq \delta$,
3. $g_3(z) \geq 0$ if $d(z,p) \geq \delta$,
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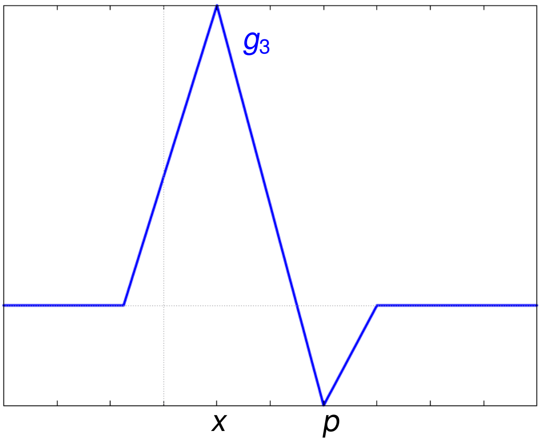

Theorem (representation of 2-local isometries)

Let X and Y be uniformly concave complete pointed metric spaces and let $\Delta\colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ be a 2-local isometry. Then there exist a subspace Y_0 of Y which is isometric to Y, a number $\lambda \in S_{\mathbb{K}}$ and a surjective a-dilation $\phi\colon Y_0 \to X$ such that

$$\Delta(f)(y_1) - \Delta(f)(y_2) = \lambda a^{-1} \left(f(\phi(y_1)) - f(\phi(y_2))\right)$$

for all $y_1, y_2 \in Y_0$ and $f \in \mathrm{Lip}_0(X)$.

Some steps of the proof

Step 1. We fix

$$\widetilde{X} = \{(x_1, x_2) \in X \times X \colon x_1 \neq x_2\},$$

$$S_{\mathbb{R}}^+ = \{1\}, \quad S_{\mathbb{C}}^+ = \left\{e^{it} \colon t \in [0, \pi[\right\},$$

and, for each $(x_1, x_2) \in \widetilde{X}$ and $f \in \mathrm{Lip}_0(X)$, we consider the set $\mathcal{B}_{(x_1,x_2),f}$ formed by the pairs $((y_1, y_2), \lambda) \in \widetilde{Y} \times S_{\mathbb{K}}$ such that

$$\frac{\Delta(f)(y_1) - \Delta(f)(y_2)}{d_Y(y_1, y_2)} = \lambda \frac{f(x_1) - f(x_2)}{d_X(x_1, x_2)},$$

and the set $\mathcal{B}_{(x_1,x_2)} = \bigcap_{f \in \mathrm{Lip}_0(X)} \mathcal{B}_{(x_1,x_2),f}$.

Then $\mathcal{B}_{(x_1,x_2)} = \mathcal{B}_{(x_1,x_2),h_{(x_1,x_2)}}$ and $\left\{\mathcal{B}_{(x_1,x_2)} \colon (x_1, x_2) \in \widetilde{X}\right\}$ is a family of nonempty subsets of $\widetilde{Y} \times S_{\mathbb{K}}$.

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Some steps of the proof

Step 2. For every $(x_1, x_2) \in \widetilde{X}$, there exist $(y_1, y_2) \in \widetilde{Y}$ and $\lambda \in S_{\mathbb{K}}^{+}$ such that

$$\mathcal{B}_{(x_1,x_2)} = \{((y_1, y_2), \lambda), ((y_2, y_1), -\lambda)\}.$$

Step 3. We define the map $\Gamma : \widetilde{X} \to \widetilde{Y}$ in the following way:

for every $(x_1, x_2) \in \widetilde{X}$, $\Gamma(x_1, x_2)$ is the unique element of $\widetilde{Y}$ for which there exists $\lambda \in S_{\mathbb{K}}^{+}$ with $(\Gamma(x_1, x_2), \lambda) \in \mathcal{B}_{(x_1,x_2)}$.

We have

- If $(y_1, y_2) = \Gamma(x_1, x_2)$, then $(y_2, y_1) = \Gamma(x_2, x_1)$.
- Γ is injective.

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- Γ is injective.

Some steps of the proof

Step 4. We define:

$$Y_0 = \left\{ y \in Y \colon (y, y_2) \in \Gamma\left(\widetilde{X}\right) \text{ for some } y_2 \in Y \right\},$$

and for each $y \in Y_0$,

$$X_y^1 = \{x_1 \in X \colon \exists\, x_2 \in X \backslash \{x_1\},\ y_2 \in Y \backslash \{y\} \text{ with } \Gamma(x_1, x_2) = (y, y_2)\},$$

$$X_y^2 = \{x_2 \in X \colon \exists\, x_1 \in X \backslash \{x_2\},\ y_2 \in Y \backslash \{y\} \text{ with } \Gamma(x_1, x_2) = (y, y_2)\}.$$

Then, for every $y \in Y_0$, either X_y^1 is a singleton or X_y^2 is a singleton.

Some steps of the proof

Step 4. We define:

$$Y_0 = \left\{ y \in Y : (y, y_2) \in \Gamma\left(\widetilde{X}\right) \text{ for some } y_2 \in Y \right\},$$

and for each $y \in Y_0$,

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Then, for every $y \in Y_0$, either X_y^1 is a singleton or X_y^2 is a singleton.

Some steps of the proof

Step 5. Let $\phi\colon Y_0 \to X$ be the map defined, for each $y \in Y_0$, by

$$\phi(y) = \begin{cases} x_1 & \text{if } X_y^1 = \{x_1\}, \\ x_2 & \text{if } X_y^2 = \{x_2\} \text{ and } X_y^1 \text{ is not a singleton.} \end{cases}$$

Then ϕ is bijective and, for all $(y_1, y_2) \in \Gamma\left(\widetilde{X}\right)$, either $\Gamma\left(\phi(y_1), \phi(y_2)\right) = (y_1, y_2)$ or $\Gamma\left(\phi(y_1), \phi(y_2)\right) = (y_2, y_1)$.

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Some steps of the proof

Step 6.
There exist numbers $a > 0$ and $\lambda \in S_{\mathbb{K}}$ such that $\phi\colon Y_0 \to X$ is an a-dilation and

$$\Delta(f)(y_1) - \Delta(f)(y_2) = \lambda a^{-1} \left(f(\phi(y_1)) - f(\phi(y_2))\right)$$

for all $y_1, y_2 \in Y_0$ and $f \in \mathrm{Lip}_0(X)$.

Step 7. Y_0 is isometric to Y.

The previous theorem can be reformulated as follows.

Corollary

Let X and Y be uniformly concave complete pointed metric spaces and let $\Delta\colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ be a 2-local isometry. Then there exist a subspace Y_0 of Y which is isometric to Y, a surjective a-dilation $\phi\colon\ Y_0 \to X$, a number $\lambda \in S_{\mathbb{K}}$ and a homogeneous Lipschitz function $\mu\colon \mathrm{Lip}_0(X) \to \mathbb{K}$ such that

$$\Delta(f)(y) = \lambda a^{-1} f(\phi(y)) + \mu(f)$$

for all $y \in Y_0$ and $f \in \mathrm{Lip}_0(X)$.

μ can be defined by

$$\mu(f) = \Delta(f)\left(\phi^{-1}(e_X)\right) \qquad (f \in \mathrm{Lip}_0(X)).$$

For a suitable choice of basepoint in Y_0, $e_{Y_0} := \phi^{-1}(e_X)$, we can see that the 2-local isometry Δ induces a surjective linear isometry.

Corollary

Let X and Y be uniformly concave complete pointed metric spaces and let $\Delta\colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ be a 2-local isometry. Then there exist an uniformly concave complete pointed metric space Y_0 such that if $R\colon \mathrm{Lip}_0(Y) \to \mathrm{Lip}_0(Y_0)$ is the restriction map given by $R(f) = f|_{Y_0}$ for all $f \in \mathrm{Lip}_0(Y)$, then $R \circ \Delta\colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y_0)$ is a surjective linear isometry.

Main theorem

Lemma

Let X be a concave metric space and let $x_1, x_2, x_3 \in X$ be three distinct points such that $d(x_1, x_2) = d(x_1, x_3)$. Given $\delta \in]0, d(x_1, x_2)[$, assume that the set

$$C = \{z \in X : d(z, x_1) \geq d(x_1, x_2),\ d(z, x_2) \geq 3\delta,\ d(z, x_3) \geq 3\delta\}$$

contains a countable subset $\{r_n : n \in \mathbb{N}\}$ of pairwise distinct points. Then there exist two Lipschitz functions $f, g : X \to \mathbb{R}$ satisfying:

i) $(f(x_1) - f(x_2))/d(x_1, x_2) = 1 = (g(x_1) - g(x_3))/d(x_1, x_3)$,

ii) $\left|f(z) - f(w)\right| / d(z, w) < 1 \quad \left((z, w) \in \widetilde{X} \setminus \{(x_1, x_2), (x_2, x_1)\}\right)$,

iii) $\left|g(z) - g(w)\right| / d(z, w) < 1 \quad \left((z, w) \in \widetilde{X} \setminus \{(x_1, x_3), (x_3, x_1)\}\right)$,

iv) $\{x \in C : (f(x), g(x)) = (f(r_n), g(r_n))\} = \{r_n\}$ for each $n \in \mathbb{N}$.

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Lemma

Let X and Y be uniformly concave complete pointed metric spaces and let $\Delta\colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ be a 2-local isometry. Let $Y_0 \subseteq Y$ be as in *representation theorem* and assume $|Y_0| \geq 3$.

If $Y_0 \neq Y$, $y \in Y \setminus Y_0$ and $y_1 \in Y_0$, then there exists a sequence $\{z_n\}$ of points in Y_0 such that

$$d_Y(z_n, y_1) = d_Y(y, y_1) \qquad (n \in \mathbb{N}),$$

$$d_Y(z_n, z_m) \geq d_Y(y, Y_0) > 0 \qquad (n, m \in \mathbb{N},\ n \neq m).$$

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Main theorem

Let X and Y be uniformly concave complete pointed metric spaces and let $\Delta\colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ be a 2-local isometry. Assume that X is also **separable**.
Then $Y_0 = Y$ and Δ is a surjective linear isometry from $\mathrm{Lip}_0(X)$ onto $\mathrm{Lip}_0(Y)$.

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