The faceless problem

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Background

All vector spaces considered will be over the reals
Let X be a vector space and C a convex subset of X

- A convex subset $F \subseteq C$ is said to be a face of C if F verifies the extremal condition with respect to C:

$$\left.\begin{array}{r} x, y \in C \\ t \in (0,1) \\ tx + (1-t)\, y \in F \end{array}\right\} \Rightarrow x, y \in F$$

- A point c of C is said to be an extreme point of C if $\{c\}$ is a face of C. We will let $\operatorname{ext}(C)$ denote the set of extreme points of C.

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It is well known that closed convex subsets with non-empty interior of Hausdorff locally convex topological vector spaces have proper faces in virtue of the Hahn-Banach Theorem.

Problem (The faceless problem)

Characterize the non-singletons convex subsets free of proper faces.

We will strongly rely on the concept of inner point.

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We will strongly rely on the concept of inner point.

Inner structure

- A point x of a non-singleton convex set C of a vector space X is said to be an inner point of C if the inner condition holds:

$$\forall c \in C \setminus \{x\} \; \exists d \in C \setminus \{x, c\} \text{ s.t. } x \in (c, d)$$

- The set of inner points of C will be denoted by $\operatorname{inn}(C)$.

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A solution to the faceless problem

Theorem

Let X be a vector space. A non-singleton convex subset C of X is free of proper faces if and only if $C = \operatorname{inn}(C)$.

One of the keys:

Remark

If x is an inner point of a convex set C, then $[x, c) \subseteq \operatorname{inn}(C)$ for all $c \in C$.

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The hint

Lemma

Let X be a vector space. Let M be a convex subset of X. If C is a convex subset of M and D is a face of M such that $\mathrm{inn}(C) \cap D \neq \varnothing$, then $C \subseteq D$.

Remark

If C is a face of M, then $C \subseteq M \setminus \mathrm{inn}(M)$.

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A first approach

Theorem

Let X be a real vector space. Let M be a convex subset of X. If F is a convex component of $M \setminus \operatorname{inn}(M)$, then F is a face of M.

Unfortunately, if $\operatorname{inn}(M) = \varnothing$, then the previous theorem does not solve the faceless problem because the only convex component of $M \setminus \operatorname{inn}(M)$ is M.

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The construction

Definition

Let X be a vector space. Let M be a convex subset of X and consider $x \in M$. We define the following sets:

- $F(x) := \bigcup\{S \subset M : S$ is a segment of M whose interior contains $x\}$.
- $C(x) := \bigcap\{S \subset M : S$ is a face of M containing $x\}$.

Theorem

Let X be a vector space. Let M be a convex subset of X and consider $x \in M$.

1. $F(x) = \varnothing$ *if and only if* $x \in \operatorname{ext}(M)$.
2. $F(x) = M$ *if and only if* $x \in \operatorname{inn}(M)$.
3. $F(x)$ *is convex if it is not empty.*
4. $F(x)$ *is a face of* M *if it is not empty.*
5. $x \notin \operatorname{ext}(M)$ *if and only if* $x \in \operatorname{inn}(F(x))$.
6. $C(x)$ *is the minimum face of* M *containing* x.
7. $F(x) = C(x)$ *if and only if* $x \notin \operatorname{ext}(M)$.
8. $x \notin \operatorname{ext}(M)$ *if and only if* $x \in \operatorname{inn}(C(x))$.
9. *If there exists a face* C *of* M *such that* $x \in \operatorname{inn}(C)$, *then* $C = C(x) = F(x)$.

The definite solution

Theorem

Let X be a vector space. A non-singleton convex subset C of X is free of proper faces if and only if $C = \mathrm{inn}(C)$.

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A characterization of linear manifolds

Corollary

Let X be a topological vector space. Let M be a non-singleton closed convex subset of X. The following conditions are equivalent:

1. *M has proper faces.*
2. *M is not a linear manifold.*

A characterization of strict convexity

Corollary

Let X be a normed space. The following conditions are equivalent:

1. *X is strictly convex.*
2. $\mathrm{inn}(C) = \varnothing$ *for all proper faces C of B_X.*

A characterization of strict convexity in transitive spaces

- A proper face $C \subseteq \mathsf{S}_X$ of the unit ball B_X of a Banach space X is said to be invariant provided that the invariance condition holds: If $T \in \mathcal{G}_X$ is a surjective linear isometry on X such that $C \subseteq T(C)$, then $C = T(C)$. Minimal and maximal faces are examples of invariant faces.
- A Banach space is called transitive if any two points of the unit sphere can be taken one into another by means of a surjective linear isometry.

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A characterization of stric convexity in transitive spaces

Lemma

Let X be a transitive Banach space. If $C \subseteq \mathsf{S}_X$ is an invariant proper face of B_X, then $\operatorname{inn}(C) = \varnothing$.

Theorem

Let X be a transitive Banach space. The following conditions are equivalent:

1. *X is strictly convex.*
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Minimal faces

Corollary

Let X be a vector space. Let M be a convex subset of X. Let D be a minimal face of M. If D is not a singleton, then $D = \operatorname{inn}(D)$.

Scholium

Let X be a topological vector space. Let M be a linearly bounded closed convex subset of X.

1. *If C is a face of M, then $C \setminus \operatorname{inn}(C) \neq \varnothing$.*
2. *If C is a minimal face of M, then C is a singleton.*

Every non-singleton linearly bounded closed convex subset of a topological vector space has proper faces.

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The Extreme Point Trichotomy

Corollary (The Extreme Point Trichotomy)

Let X be a vector space. Let M be a convex subset of X. Let $x \in M$. There are only three disjoint possibilities for x:

1. *x is an extreme point of M.*
2. *x is an inner point of M.*
3. *x is an inner point of a proper face of M.*

Another problem of similar nature

Problem

Characterize when a proper convex subset of a convex set is contained in a proper face.

Known situations:

- If M is non-singleton and convex and $x \in \mathrm{ext}(M)$, then $M \setminus \{x\}$ is trivially a proper convex subset of M not contained in a proper face of M.
- If $\mathrm{inn}(M) \neq \varnothing$, then every convex set $N \subseteq M \setminus \mathrm{inn}(M)$ is contained in a convex component of $M \setminus \mathrm{inn}(M)$ which is a proper face of M.

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Generalizing the previous construction

Definition

Let X be a vector space. Let M be a convex subset of X and consider a convex subset N of M. We define the following sets:

- $F(N) := \bigcup_{n \in N} F(n)$.
- $C(N) := \bigcap \{S \subset M : S \text{ is a face of } M \text{ containing } N\}$.

Theorem

Let X be a vector space. Let $N \subseteq M \subseteq X$ be convex subsets.

1. *$F(N) = \varnothing$ iff N consists only of one extreme point of M.*
2. *If $\operatorname{inn}(M) \neq \varnothing$, then $F(N) = M$ iff $N \cap \operatorname{inn}(M) \neq \varnothing$.*
3. *$F(N)$ is convex if it is not empty.*
4. *$F(N)$ is a face of M if it is not empty.*
5. *$C(N)$ is the minimum face of M containing N.*
6. *If $N \setminus \operatorname{ext}(M) \neq \varnothing$ and D is a face of M containing $N \setminus \operatorname{ext}(M)$, then D contains N.*
7. *If $N \setminus \operatorname{ext}(M) \neq \varnothing$, then $F(N) = F(N \setminus \operatorname{ext}(M))$ and $C(N \setminus \operatorname{ext}(M)) = C(N)$.*
8. *$F(N) = C(N)$ if and only if $N \setminus \operatorname{ext}(M) \neq \varnothing$.*

The solution to the other problem

Corollary

Let X be a vector space. Let M be a convex subset of X with $\mathrm{inn}(M) \neq \varnothing$. A proper convex subset N of M is contained in a proper face of M if and only if $N \subseteq M \setminus \mathrm{inn}(M)$.

The previous corollary fails if we drop the hypothesis that $\mathrm{inn}(M) \neq \varnothing$.

Corollary

Let X be a vector space. Let M be a non-singleton convex subset of X. If $\mathrm{ext}(M) \neq \varnothing$, then $M \setminus \mathrm{ext}(M)$ is a non-empty proper convex subset of M not contained in a proper face of M.

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The finite dimensional case

Theorem

Let X be a finite dimensional Banach space. Let M be a convex subset of X. If C is a face of M, then C is closed in M.

The infinite dimensional case

Theorem

Let X be an infinite dimensional Banach space. There exists an absolutely convex and absorbing subset M of X with a proper face C such that C is dense in M and $\operatorname{inn}(C) = C$. In particular, C is a non-closed proper face of M whose closure in M is not a proper face of M.

Proof sketch. It suffices to consider $Y := \ker(g)$ where $g : X \to \mathbb{R}$ is linear but not continuous, $x \in X$ such that $g(x) = 1$, $M := \operatorname{co}(Y \cup \{x, -x\})$ and $C := Y$. $\square$

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Theorem

Let X be an infinite dimensional Banach space. There exists a bounded, closed, and absolutely convex subset M of X with two proper faces C and D such that:

1. *$C \subsetneq D$, C is dense in D, and D is not dense in M. In particular, C is not closed.*
2. $\operatorname{inn}(C) = \operatorname{inn}(D) = \varnothing$.

Moreover, if X contains an isomorphic copy of c_0 or ℓ_p for $1 < p < \infty$, then C and D can be chosen so that their closures are not faces of M.

Proof sketch. Let $(e_n)_{n\in\mathbb{N}} \subset \mathsf{S}_X$ be a basic sequence and consider

$$M := \left\{ \sum_{n=1}^{\infty} t_n e_n : (t_n)_{n\in\mathbb{N}} \in \mathsf{B}_{\ell_1} \right\}$$

$$C := \left\{ \sum_{n=1}^{\infty} t_n e_n : t_n \geq 0, \sum_{n=1}^{\infty} t_n = 1, (t_n)_{n\in\mathbb{N}} \in c_{00} \right\}$$

and

$$D := \left\{ \sum_{n=1}^{\infty} t_n e_n : t_n \geq 0, \sum_{n=1}^{\infty} t_n = 1 \right\}.$$

$\square$

Theorem

Let X be an infinite dimensional Banach space. There exists a bounded convex subset D of X containing proper faces but free of inner points. Moreover, if X contains an isomorphic copy of ℓ_1, then D can be chosen to be closed.

Proof sketch. Let $(e_n)_{n\in\mathbb{N}} \subset \mathsf{S}_X$ be a basic sequence. We may assume without loss of generality that e_n is an extreme point of $\mathsf{B}_{\overline{\mathrm{span}}\{e_n : n\in\mathbb{N}\}}$ for all $n \in \mathbb{N}$. Now take

$$D := \left\{ \sum_{n=1}^{\infty} t_n e_n : t_n \geq 0, \sum_{n=1}^{\infty} t_n = 1 \right\}$$

$\square$

Example

Let X be an infinite dimensional Banach space containing an isomorphic copy of ℓ_1. Let $(e_n)_{n\in\mathbb{N}} \subset \mathsf{S}_X$ be the image of the ℓ_1-basis and assume that e_n is an extreme point of $\mathsf{B}_{\overline{\mathrm{span}}\{e_n : n\in\mathbb{N}\}}$ for all $n \in \mathbb{N}$. Consider the bounded, closed, convex set

$$M := \left\{ \sum_{n=1}^{\infty} t_n e_n : t_n \geq 0, \sum_{n=1}^{\infty} t_n = 1 \right\}.$$

We know that $\mathrm{inn}(M) = \varnothing$. Now take $N := M \setminus \{e_1\}$. Note that N is a proper convex subset of M not contained in any proper face of M.

Theorem

Every infinite dimensional Banach space can be equivalently renormed so that its unit ball contains a non-closed face.

Corollary

Every infinite dimensional Banach space containing an isomorphic copy of c_0 or ℓ_p, $1 < p < \infty$, can be equivalently renormed so that its unit ball contains a face whose closure is not a face.

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THANK YOU FOR YOR ATTENTION!