Stabilization of switched linear systems

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University of Cádiz

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Contents

Introduction to control theory

Control system

In a control system can be several types of variables:

- **time** variable

Introduction to control theory

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- **state** variables

Introduction to control theory

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Introduction to control theory

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- **measured** variables

Introduction to control theory

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Introduction to control theory

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Introduction to control theory

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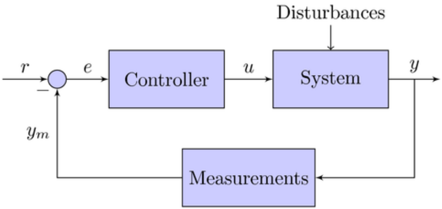

Figure: Block diagram of general control system

Introduction to control theory

Objective

Design a control signal which achieves some **property** or feature desired for the state variable for every noise/uncertain signal.

Introduction to control theory

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To control the state variable

Introduction to control theory

Properties for control problems

It can be studied different properties:

- **Stability**

Introduction to control theory

Properties for control problems

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- **Stabilization**

Properties for control problems

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Introduction to control theory

Properties for control problems

It can be studied different properties:

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- **Invariant sets**

Introduction to control theory

Properties for control problems

It can be studied different properties:

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- **Optimization**

Introduction to control theory

Relation between variables

A control system can be given by:

- **ODE** (ordinary differential equation)

Introduction to control theory

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- **PDE** (partial differential equation)

Introduction to control theory

Relation between variables

A control system can be given by:

- **ODE** (ordinary differential equation)
- **PDE** (partial differential equation)
- **Difference equation**

Introduction to control theory

Relation between variables

A control system can be given by:

- **ODE** (ordinary differential equation)
- **PDE** (partial differential equation)
- **Difference equation**
- A mixture: for instance, a **coupled ODE-PDE** system

Introduction to control theory

Classification of control systems

- **Discrete systems**: variables are discrete

Introduction to control theory

Classification of control systems

- **Discrete systems**: variables are discrete
- **Continuous systems**: variables are continuous

Introduction to control theory

Classification of control systems

- **Discrete systems**: variables are discrete
- **Continuous systems**: variables are continuous
- **Hybrid systems**: variables are discrete and continuous

Introduction to control theory

Classification of control

- **Open loop control**: control depends on time

Introduction to control theory

Classification of control

- **Open loop control**: control depends on time
- **Closed loop control (feedback)**: control depends on state

Switched linear control systems

Switched linear control system

A switched linear control system is given by

$$\dot{x}(t) = A_{\sigma(t)}x(t),$$

where A_i are real matrices for each $i \in I$.

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- The function $\sigma : \mathbb{R}_+ \to I$ is called **switching signal**.

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- The function $\sigma : \mathbb{R}_+ \to I$ is called **switching signal**.
- σ is a piecewise constant function.
- The discontinuities of σ are called **switching times**.

Switched linear control systems

Example of switching signal

$$\sigma(t) = \begin{cases} 1, & \text{if } t \in [0, 1) \\ 2, & \text{if } t \in [1, 10) \\ 1, & \text{if } t \in [10, 13.4) \\ 2, & \text{if } t \in [13.4, 20) \\ \vdots & \end{cases}$$

Switched linear control systems

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Switched linear control systems

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) \\ x(0) = x_0 \end{cases} \tag{1}$$

Switched linear control systems

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- For an initial condition $x_0 \in \mathbb{R}^n$ and a switching signal $\sigma \in \mathcal{S}(\mathbb{R}_+, I)$, the solution of (3) is denote by $\varphi(\cdot; x_0, \sigma)$.

Switched linear control systems

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A switched linear control system is given by

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where A_k is a real matrices for each $k \in I$.

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Problem 1

If every subsystem A_k is unstable.
Construct a switching signal such that the system is stable.

Switched linear control systems

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Problem 2

If every subsystem A_k is stable.
Is it the system stable for every switching signal?

Switched linear control systems

σ-convergent

Let σ be a switching signal. A point $x_0 \in \mathbb{R}^n$ is σ-convergent for the switched system (3) if

$$\lim_{t \to +\infty} \varphi(t; x_0, \sigma) = 0$$

Switched linear control systems

σ-convergent

Let σ be a switching signal. A point $x_0 \in \mathbb{R}^n$ is σ-convergent for the switched system (3) if

$$\lim_{t \to +\infty} \varphi(t; x_0, \sigma) = 0$$

Convergent

A point $x_0 \in \mathbb{R}^n$ is convergent for the switched system (3) if there is a switching signal σ such that x_0 is σ-convergent.

Stabilization of second order switched linear systems

Example 1

$$A_1 = \begin{pmatrix} .1 & -2 \\ .5 & .1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} .1 & -.5 \\ 2 & .1 \end{pmatrix}$$

Stabilization of second order switched linear systems

Example 1

$$A_1 = \begin{pmatrix} .1 & -2 \\ .5 & .1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} .1 & -.5 \\ 2 & .1 \end{pmatrix}$$

- Eigenvalues of A_1: $.1 \pm i$
- Eigenvalues of A_2: $.1 \pm i$

Stabilization of second order switched linear systems

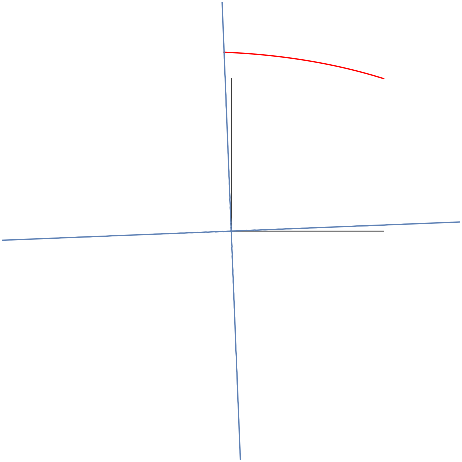
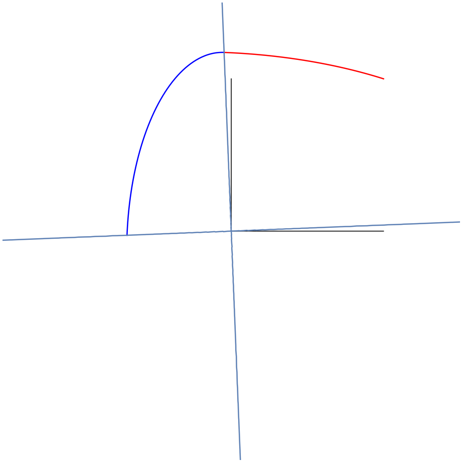


Figure: Solution for example 1

Stabilization of second order switched linear systems



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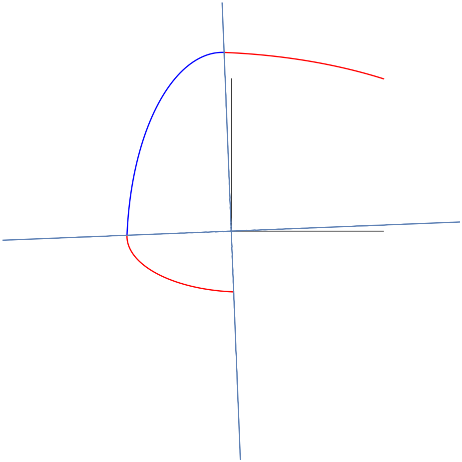

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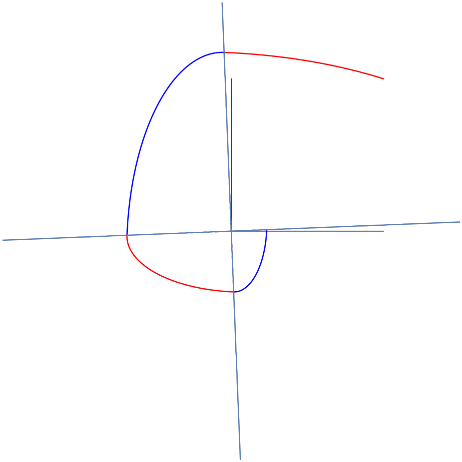

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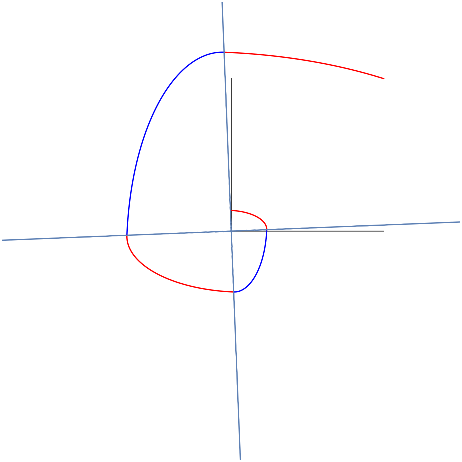

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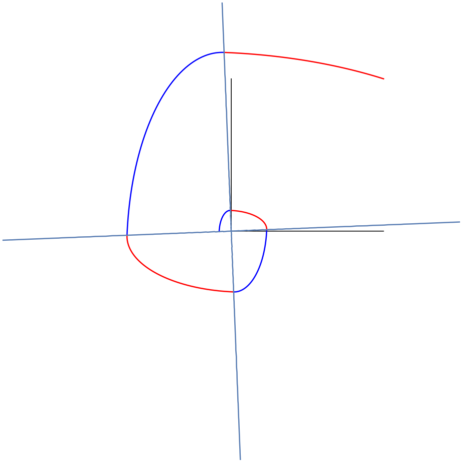

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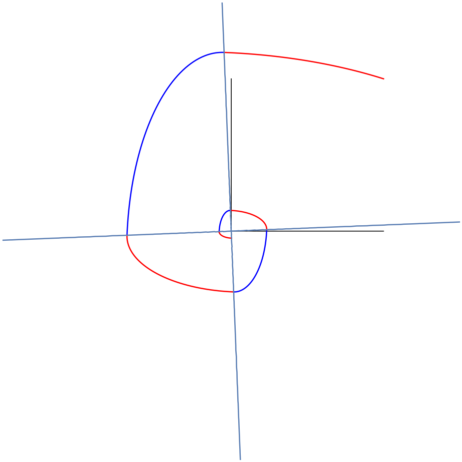


Figure: Solution for example 1

Stabilization of second order switched linear systems

Example 2

$$A_1 = \begin{pmatrix} -.1 & -2 \\ .5 & -.1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} -.1 & -.5 \\ 2 & -.1 \end{pmatrix}$$

Stabilization of second order switched linear systems

Example 2

$$A_1 = \begin{pmatrix} -.1 & -2 \\ .5 & -.1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} -.1 & -.5 \\ 2 & -.1 \end{pmatrix}$$

- Eigenvalues of A_1: $-.1 \pm i$
- Eigenvalues of A_2: $-.1 \pm i$

Stabilization of second order switched linear systems

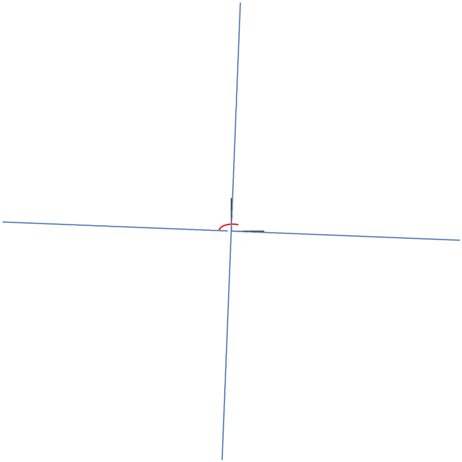
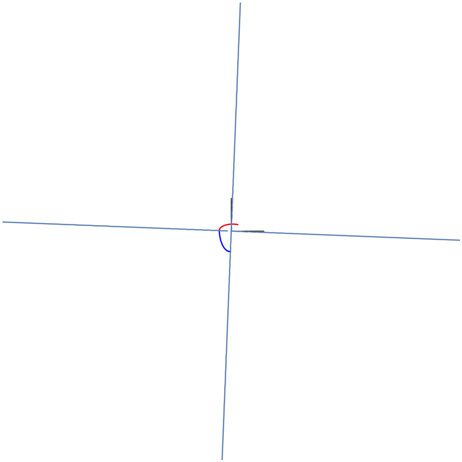


Figure: Solution for example 2

Stabilization of second order switched linear systems



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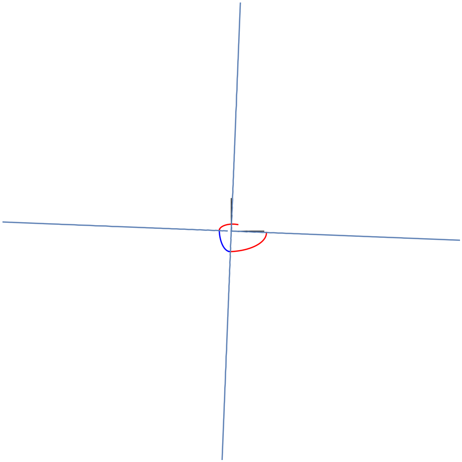

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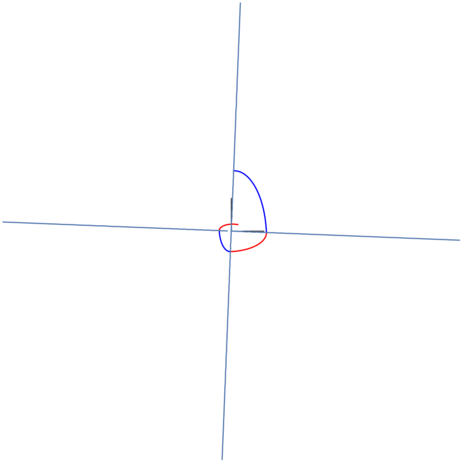

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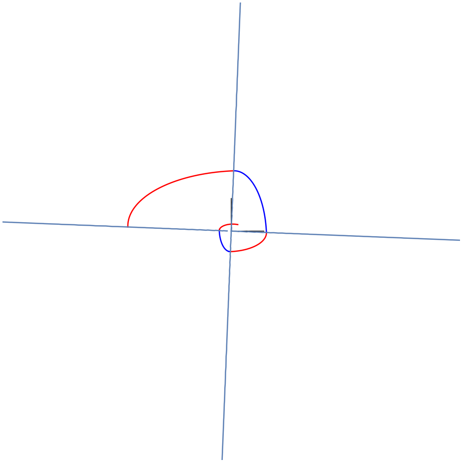

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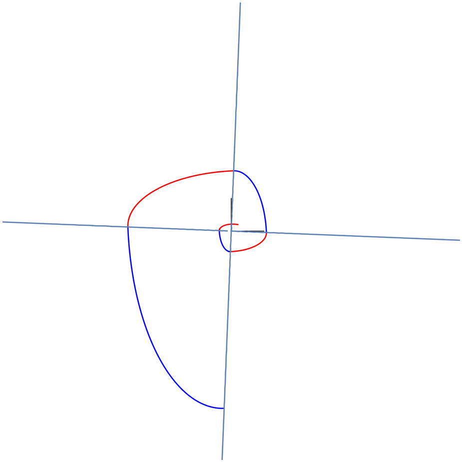

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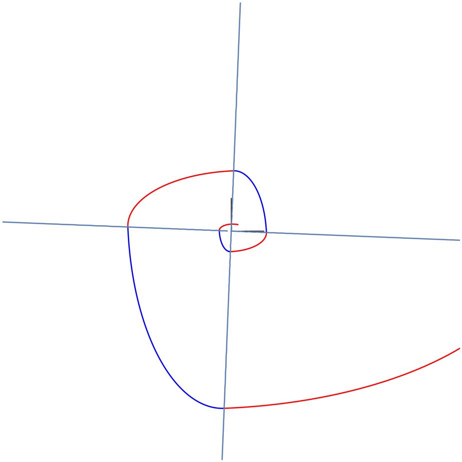

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Stabilization of second order switched linear systems

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) \\ x(0) = x_0 \end{cases} \quad (2)$$

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Stabilization of switched linear systems

For a switched linear system, the stabilization problem is the classification of convergent point.

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For a switched linear system, the stabilization problem is the classification of convergent point.

- For second order switched linear systems with two subsystems the stabilization problem is solved.

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[Benítez et al., 2011] F. Benítez and C. Pérez.
Methods of stabilizing or destabilizing a switched linear system,
Journal of Mathematical Sciences, vol. 177, no. 3, pp. 345–356, 2011.

Invariant set for third order switched systems

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) \\ x(0) = x_0 \end{cases} \quad (3)$$

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Invariant set for switched systems

A set $S \subset \mathbb{R}^n$ is an invariant set for the switched system if there exist a switching law σ such that if $x_0 \in S$ then $\phi(t; x_0, \sigma) \in S$ for each $t \geq 0$.

Invariant set for third order switched systems

$$\dot{x} = A_k x, \qquad k = 1, 2, 3,$$

where A_1, A_2, A_3 are 3×3 matrices.

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Assumption 1

Each A_k, $k = 1, 2, 3$, has complex eigenvalues,i.e. the numbers $\lambda_k, a_k + b_k i, a_k - b_k i$ are the eigenvalues of A_k with $b_k \neq 0$.

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Assumption 2

Let $v_k \in \mathbb{R}^3$ be an eigenvector of A_k associated to the real eigenvalue λ_k, $k = 1, 2, 3$, i.e. v_k is a non-zero vector such that $A_k v_k = \lambda_k v_k$, then v_1, v_2, v_3 are linear independent vectors.

Invariant set for third order switched systems

Proposition

Let P be a $n \times n$ non-singular matrix. The following statements are equivalent

1. $P(S) \subset \mathbb{R}^n$ is an invariant set for the switched system
$$\dot{x} = A_\sigma x,$$
2. $S \subset \mathbb{R}^n$ is an invariant set for the switched system
$$\dot{y} = P^{-1} A_\sigma P y.$$

Where we denote $P(S) = \{Py : y \in S\}$.

Invariant set for third order switched systems

We denote $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

Assumption 1

Each A_k, $k = 1, 2, 3$, has complex eigenvalues,i.e. the numbers $\lambda_k, a_k + b_k i, a_k - b_k i$ are the eigenvalues of A_k with $b_k \neq 0$.

Assumption 2'

For each $k = 1, 2, 3$ the vector e_k is an eigenvector of A_k associated to the eigenvalue λ_k.

Invariant set for third order switched systems

Proposition

If A_1, A_2, A_3 verifies Assumption 1 and 2' then

$$A_1 = \begin{pmatrix} \lambda_1 & a^1_{12} & a^1_{13} \\ 0 & a^1_{22} & a^1_{23} \\ 0 & a^1_{32} & a^1_{33} \end{pmatrix} \qquad A_2 = \begin{pmatrix} a^2_{11} & 0 & a^2_{13} \\ a^2_{21} & \lambda_2 & a^2_{23} \\ a^2_{31} & 0 & a^2_{33} \end{pmatrix}$$

$$A_3 = \begin{pmatrix} a^3_{11} & a^3_{12} & 0 \\ a^3_{21} & a^3_{22} & 0 \\ a^3_{31} & a^3_{32} & \lambda_3 \end{pmatrix}$$

with $a^1_{23}a^1_{32} < 0$, $a^2_{13}a^2_{31} < 0$ and $a^3_{12}a^3_{21} < 0$.

Invariant set for third order switched systems

Octants of $\mathbb{R}^3$

Every octant of $\mathbb{R}^3$ is identified with three signs, i.e. each $(a, b, c) \in \{-1, +1\}^3$ is identified with the octant

$$O(a, b, c) = \{(x_1, x_2, x_3) : ax_1 \geq 0, bx_2 \geq 0, cx_3 \geq 0\}.$$

Invariant set for third order switched systems

Octants of $\mathbb{R}^3$

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$$O(a, b, c) = \{(x_1, x_2, x_3) : ax_1 \geq 0, bx_2 \geq 0, cx_3 \geq 0\}.$$

The faces of $O(a, b, c)$ are

$$\begin{aligned}
O(0, b, c) &= \{(0, x_2, x_3) : bx_2 > 0, cx_3 > 0\},\\
O(a, 0, c) &= \{(x_1, 0, x_3) : ax_1 > 0, cx_3 > 0\},\\
O(a, b, 0) &= \{(x_1, x_2, 0) : ax_1 > 0, bx_2 > 0\},\\
O(a, 0, 0) &= \{(x_1, 0, 0) : ax_1 > 0\},\\
O(0, b, 0) &= \{(0, x_2, 0) : bx_2 > 0\},\\
O(0, 0, c) &= \{(0, 0, x_3) : cx_3 > 0\}.
\end{aligned}$$

Invariant set for third order switched systems

Face switching law

$$\sigma(x) = \begin{cases} s_1 & \text{if } x \in O(0,b,c) \\ s_2 & \text{if } x \in O(a,0,c) \\ s_3 & \text{if } x \in O(a,b,0) \\ s_{12} & \text{if } x \in O(0,0,c) \\ s_{13} & \text{if } x \in O(0,b,0) \\ s_{23} & \text{if } x \in O(a,0,0) \end{cases}$$

where $s_1, s_2, s_3 \in \{1,2,3\}$ and $s_{ij} = s_i$ or $s_{ij} = s_j$ for each $1 \leq i < j \leq 3$.

Invariant set for third order switched systems

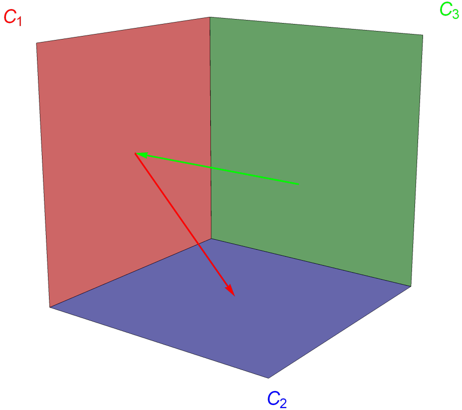

Figure: Switching when the state reaches the face

Invariant set for third order switched systems

Face switching laws σ_1 and σ_2

$$\sigma_1(x) = \begin{cases} 3 & \text{if } x \in O(0,b,c) \\ 1 & \text{if } x \in O(a,0,c) \\ 2 & \text{if } x \in O(a,b,0) \\ 3 & \text{if } x \in O(0,0,c) \\ 2 & \text{if } x \in O(0,b,0) \\ 1 & \text{if } x \in O(a,0,0) \end{cases} \qquad \sigma_2(x) = \begin{cases} 2 & \text{if } x \in O(0,b,c) \\ 3 & \text{if } x \in O(a,0,c) \\ 1 & \text{if } x \in O(a,b,0) \\ 3 & \text{if } x \in O(0,0,c) \\ 2 & \text{if } x \in O(0,b,0) \\ 1 & \text{if } x \in O(a,0,0) \end{cases}$$

Invariant set for third order switched systems

Theorem

Let $O(a, b, c)$ be an octant in $\mathbb{R}^3$ with $(a, b, c) \in \{-1, +1\}^3$, then the following statements are equivalent

1. $O(a, b, c)$ is an invariant set for the switched system with the face switching law σ,
2. the following statements hold
   1. $ab\, e_1' A_{s_1} e_2 \geq 0$ and $ac\, e_1' A_{s_1} e_3 \geq 0$,
   2. $ab\, e_2' A_{s_2} e_1 \geq 0$ and $bc\, e_2' A_{s_2} e_3 \geq 0$,
   3. $ac\, e_3' A_{s_3} e_1 \geq 0$ and $bc\, e_3' A_{s_3} e_2 \geq 0$,
   4. $bc\, e_2' A_{s_1} e_3 \geq 0$ or $ac\, e_1' A_{s_2} e_3 \geq 0$,
   5. $ab\, e_1' A_{s_3} e_2 \geq 0$ or $bc\, e_3' A_{s_1} e_2 \geq 0$,
   6. $ab\, e_2' A_{s_3} e_1 \geq 0$ or $ac\, e_3' A_{s_2} e_1 \geq 0$.

   Where $'$ denotes transpose.

Invariant set for third order switched systems

Corollary for σ_1

If A_1, A_2 and A_3 verify Assumption 1 and 2', the following statements are equivalent

1. $O(a, b, c)$ is invariant for the switched system with face switching lawσ_1,
2. $\text{sign}(a_{12}^3) = \text{sign}(ab)$, $\text{sign}(a_{23}^1) = \text{sign}(bc)$ and $\text{sign}(a_{31}^2) = \text{sign}(ac)$.

Invariant set for third order switched systems

Corollary for σ_1

If A_1, A_2 and A_3 verify Assumption 1 and 2', the following statements are equivalent

1. $O(a, b, c)$ is invariant for the switched system with face switching lawσ_1,
2. $\text{sign}(a_{12}^3) = \text{sign}(ab)$, $\text{sign}(a_{23}^1) = \text{sign}(bc)$ and $\text{sign}(a_{31}^2) = \text{sign}(ac)$.

Corollary for σ_2

If A_1, A_2 and A_3 verify Assumption 1 and 2', the following statements are equivalent

1. $O(a, b, c)$ is invariant for the switched system with face switching lawσ_2,
2. $\text{sign}(a_{12}^3) = -\text{sign}(ab)$, $\text{sign}(a_{23}^1) = -\text{sign}(bc)$ and $\text{sign}(a_{31}^2) = -\text{sign}(ac)$.

Invariant set for third order switched systems

			σ_1			σ_2		
a^1_{23}	a^2_{31}	a^3_{12}	bc	ac	ab	bc	ac	ab
$+$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$
$+$	$+$	$-$	$+$	$+$	$-$	$-$	$-$	$+$
$+$	$-$	$+$	$+$	$-$	$+$	$-$	$+$	$-$
$+$	$-$	$-$	$+$	$-$	$-$	$-$	$+$	$+$
$-$	$+$	$+$	$-$	$+$	$+$	$+$	$-$	$-$
$-$	$+$	$-$	$-$	$+$	$-$	$+$	$-$	$+$
$-$	$-$	$+$	$-$	$-$	$+$	$+$	$+$	$-$
$-$	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$+$

Table: Signs of bc, ac and ab deduce from signs of a^1_{23}, a^2_{31} and a^3_{12}, for each law σ_1 and σ_2.

Invariant set for third order switched systems

a	b	c	bc	ac	ab
$+$	$+$	$+$	$+$	$+$	$+$
$+$	$+$	$-$	$-$	$-$	$+$
$+$	$-$	$+$	$-$	$+$	$-$
$+$	$-$	$-$	$+$	$-$	$-$
$-$	$+$	$+$	$+$	$-$	$-$
$-$	$+$	$-$	$-$	$+$	$-$
$-$	$-$	$+$	$-$	$-$	$+$
$-$	$-$	$-$	$+$	$+$	$+$

Table: All posibilities of signs of bc, ac and ab for each octant $O(a, b, c)$.

Invariant set for third order switched systems

			σ_1			σ_2		
a^1_{23}	a^2_{31}	a^3_{12}	bc	ac	ab	bc	ac	ab
$+$	$+$	$+$	$+$	$+$	$+$			
$+$	$+$	$-$				$-$	$-$	$+$
$+$	$-$	$+$				$-$	$+$	$-$
$+$	$-$	$-$	$+$	$-$	$-$			
$-$	$+$	$+$				$+$	$-$	$-$
$-$	$+$	$-$	$-$	$+$	$-$			
$-$	$-$	$+$	$-$	$-$	$+$			
$-$	$-$	$-$				$+$	$+$	$+$

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Step 2. Calculating $a, b, c \in \{-1, +1\}^3$ such that the octant $O(a, b, c)$ is invariant for the switched system with matrices $P^{-1}A_1P, P^{-1}A_2P, P^{-1}A_3P$ and the face switching law σ_1 or σ_2.

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Step 3. By a previously Proposition, the set

$$P(O(a, b, c)) = \{Px : x \in O(a, b, c)\}$$

is invariant for the original switched system with matrices A_1, A_2 and A_3.

Invariant set for third order switched systems

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{10}{3} \\ 0 & \frac{4}{3} & \frac{5}{3} \end{pmatrix}, \; A_2 = \begin{pmatrix} -3 & 2 & 2 \\ 0 & -1 & 0 \\ -4 & 4 & 1 \end{pmatrix}, \; A_3 = \begin{pmatrix} 4 & -2 & 0 \\ 4 & 0 & 0 \\ 1 & -\frac{3}{2} & 1 \end{pmatrix}.$$

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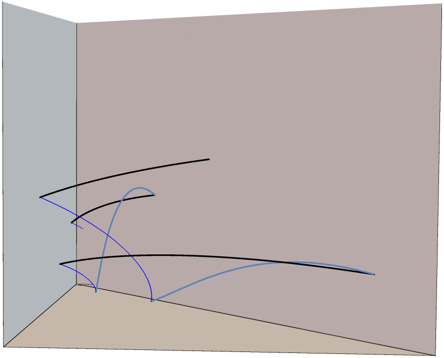

Stabilization of third order switched linear systems

Proposition

Let P be a non-singular matrix. The initial condition $x_0 \in \mathbb{R}^n$ is a σ-convergent point for the switched system

$$\dot{x}(t) = A_{\sigma(t)}x(t),$$

if and only if the initial condition $y_0 = P^{-1}x_0$ is a σ-convergent point for the switched system

$$\dot{y}(t) = P^{-1}A_{\sigma(t)}Py_0.$$

Stabilization of third order switched linear systems

We denote $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

Assumption 1

Each A_k, $k = 1, 2, 3$, has complex eigenvalues,i.e. the numbers $\lambda_k, a_k + b_k i, a_k - b_k i$ are the eigenvalues of A_k with $b_k \neq 0$.

Assumption 2'

For each $k = 1, 2, 3$ the vector e_k is an eigenvector of A_k associated to the eigenvalue λ_k.

- Let the positive octant

$$C = \{(x_1, x_2, x_3) : x_1, x_2, x_3 \geq 0\} \tag{4}$$

Stabilization of third order switched linear systems

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- Define the faces

$$C_1 = \{(x_1, 0, x_3) : x_1, x_3 \geq 0\}, \quad C_2 = \{(x_1, x_2, 0) : x_1, x_2 \geq 0\}$$
$$\text{and} \quad C_3 = \{(0, x_2, x_3) : x_2, x_3 \geq 0\}$$

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- Define the edge

$$V_1 = \{(x_1, 0, 0) : x_1 \geq 0\}, \quad V_2 = \{(0, x_2, 0) : x_2 \geq 0\}$$
$$\text{and} \quad V_3 = \{(0, 0, x_3) : x_3 \geq 0\}$$

Stabilization of third order switched linear systems

Face switching laws σ_1

$$\sigma_1(x) = \begin{cases} 1 & \text{if } x \in C_1 \\ 2 & \text{if } x \in C_2 \\ 3 & \text{if } x \in C_3 \\ 1 & \text{if } x \in V_1 \\ 2 & \text{if } x \in V_2 \\ 3 & \text{if } x \in V_3 \end{cases}$$

Stabilization of third order switched linear systems

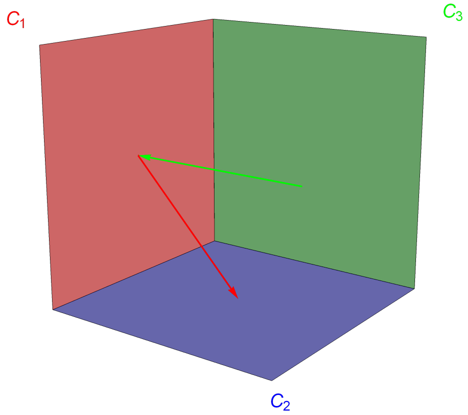

Figure: Switching when the state reaches the face

Stabilization of third order switched linear systems

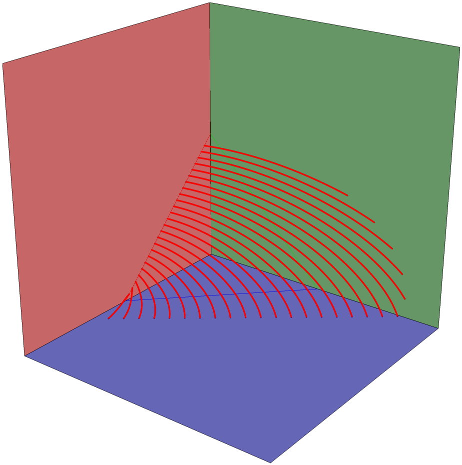

Figure: From C_1 face to C_2 and C_3 face

Stabilization of third order switched linear systems

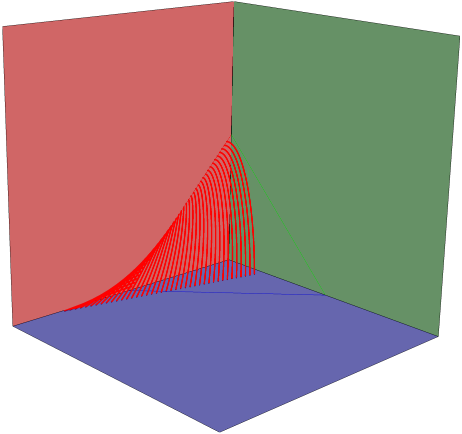

Figure: From C_1 face to C_2

Stabilization of third order switched linear systems

- $\Phi_1 : C_1 \to C_2$,

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$$\Phi = \Phi_3 \circ \Phi_2 \circ \Phi_1 : C_1 \to C_1.$$

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Frobenius theorem

An irreducible non-negative matrix M always has a positive eigenvalue r that is a simple root of the characteristic equation, there is an eigenvector of r with positive coordinates, and the other eigenvalues has modulus less or equals than r.

Stabilization of third order switched linear systems

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An irreducible non-negative matrix M always has a positive eigenvalue r that is a simple root of the characteristic equation, there is an eigenvector of r with positive coordinates, and the other eigenvalues has modulus less or equals than r.

Consider M the matrix $e^{A_3 T_3} e^{A_2 T_2} e^{A_1 T_1}$ with respect the basis $\{e_1, e_3\}$.

Stabilization of third order switched linear systems

Theorem

If the previous eigenvalue r of M is less than 1, then every initial condition $x_0 \in C$ is σ_1-convergent.

Example of stability on third order switched linear systems

- $A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -25 & 1 \end{pmatrix}$
- $A_2 = \begin{pmatrix} 1 & 0 & -10 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$
- $A_3 = \begin{pmatrix} 1 & 1 & 0 \\ -10 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

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- $A_3 = \begin{pmatrix} 1 & 1 & 0 \\ -10 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

- Eigenvalues of A_1: $1, 1 \pm 5$.
- Eigenvalues of A_2: $1, 1 \pm 3.16228$.
- Eigenvalues of A_3: $1, 1 \pm 3.16228$.

Example of stability on third order switched linear systems

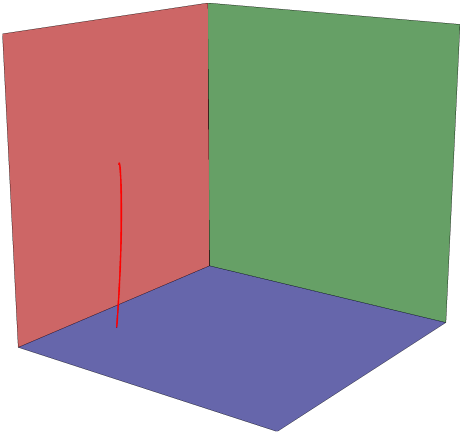
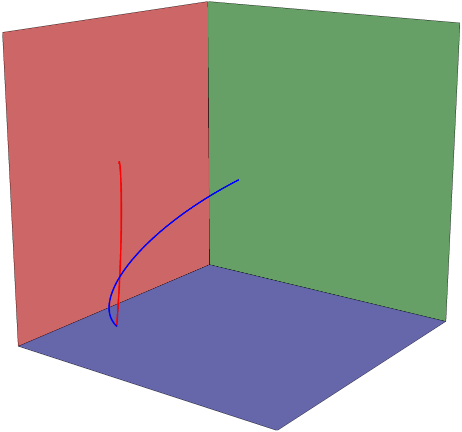


Figure: Step of switched signal σ_1 for the example

Example of stability on third order switched linear systems



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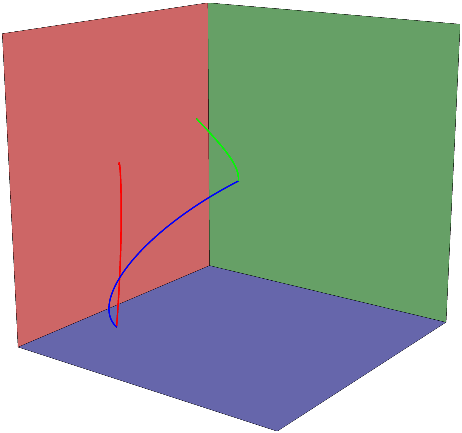

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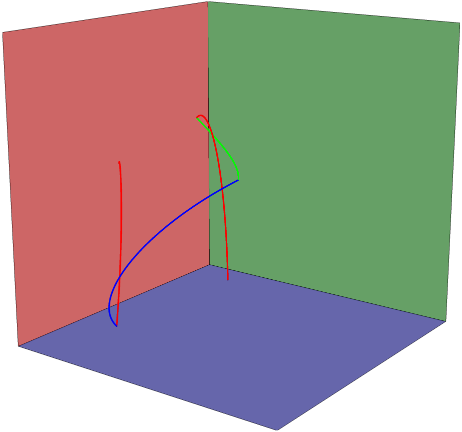

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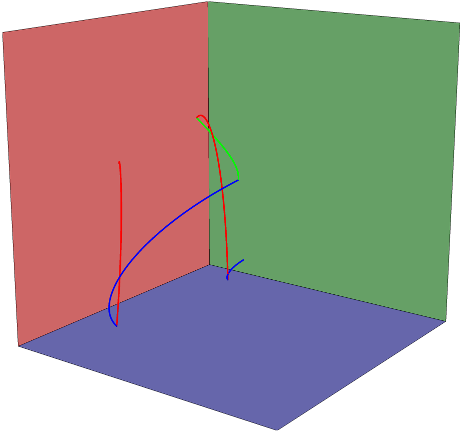

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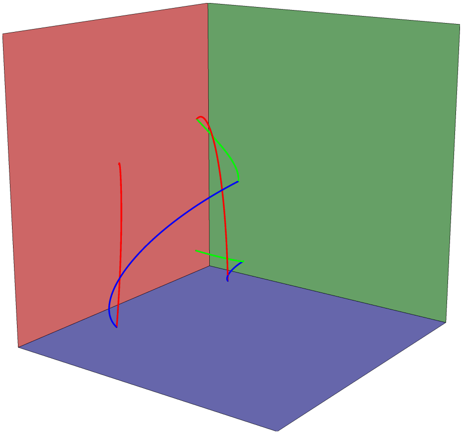

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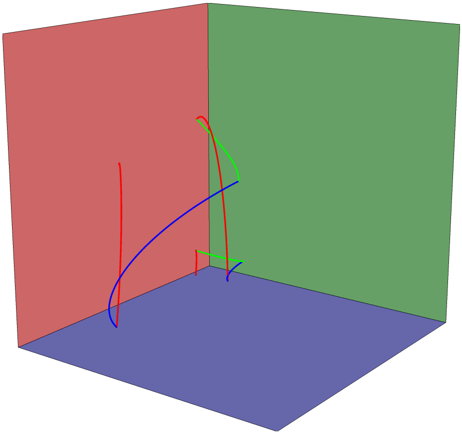

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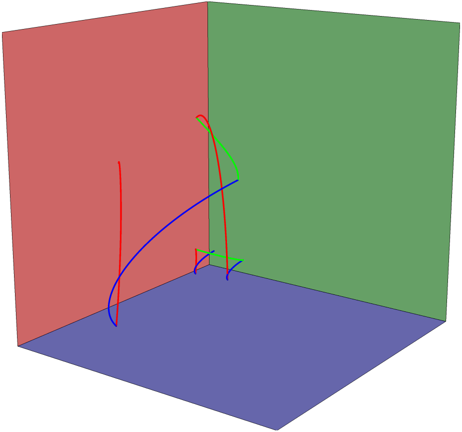

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Thank for your attention