Combined use of λ-symmetries and solvable structures for finding explicit exact solutions of ordinary differential equations

A. Ruiz, C. Muriel

University of Cádiz,
Department of Mathematics
adrian.ruiz@uca.es

I Workshop on functional Analysis

September 5, 2018

Table of contents

Lie symmetries and Lie symmetry algebras

- Let us consider an nth-order ODE

$$x_n = F(t, x, \ldots, x_{n-1}) \tag{1}$$

defined for $(t, x) \in M$, being $M \subset \mathbb{R}^2$ some open and simply connected subset.

Lie symmetries and Lie symmetry algebras

- Let us consider an nth-order ODE

$$x_n = F(t, x, \dots, x_{n-1}) \tag{1}$$

defined for $(t, x) \in M$, being $M \subset \mathbb{R}^2$ some open and simply connected subset.

- t and x are the independent and dependent variables, respectively, and $x_i = \frac{d^i x}{dt^i}$ for $i = 1, \dots, n$.

Lie symmetries and Lie symmetry algebras

- Let us consider an nth-order ODE

$$x_n = F(t, x, \dots, x_{n-1}) \tag{1}$$

defined for $(t, x) \in M$, being $M \subset \mathbb{R}^2$ some open and simply connected subset.

- t and x are the independent and dependent variables, respectively, and $x_i = \frac{d^i x}{dt^i}$ for $i = 1, \dots, n$.
- $M^{(n)}$ stands for the nth-order jet space, which has local coordinates $(t, x^{(n)}) = (t, x, x_1, \dots, x_n)$.

Lie symmetries and Lie symmetry algebras

- Let us consider an nth-order ODE

$$x_n = F(t, x, \dots, x_{n-1}) \tag{1}$$

defined for $(t, x) \in M$, being $M \subset \mathbb{R}^2$ some open and simply connected subset.
- t and x are the independent and dependent variables, respectively, and $x_i = \frac{d^i x}{dt^i}$ for $i = 1, \dots, n$.
- $M^{(n)}$ stands for the nth-order jet space, which has local coordinates $(t, x^{(n)}) = (t, x, x_1, \dots, x_n)$.
- The total derivative operator

$$D_t = \partial_t + x_1 \partial_x + x_2 \partial_{x_1} + \cdots + x_n \partial_{x_{n-1}} + \cdots$$

Lie symmetries and Lie symmetry algebras

- Let us consider an nth-order ODE

$$x_n = F(t, x, \ldots, x_{n-1}) \tag{1}$$

defined for $(t, x) \in M$, being $M \subset \mathbb{R}^2$ some open and simply connected subset.

- t and x are the independent and dependent variables, respectively, and $x_i = \frac{d^i x}{dt^i}$ for $i = 1, \ldots, n$.
- $M^{(n)}$ stands for the nth-order jet space, which has local coordinates $(t, x^{(n)}) = (t, x, x_1, \ldots, x_n)$.
- The total derivative operator

$$D_t = \partial_t + x_1 \partial_x + x_2 \partial_{x_1} + \cdots + x_n \partial_{x_{n-1}} + \cdots$$

- Vector field associated to equation (1):

$$\mathbf{A} = \partial_t + x_1 \partial_x + x_2 \partial_{y_1} + \cdots + F(t, x, \ldots, x_{n-1}) \partial_{x_{n-1}}. \tag{2}$$

Definition (standard prolongation of vector fields)

Let $\mathbf{X} = \xi(t,x)\partial_t + \eta(t,x)\partial_x$ be a smooth vector field defined on M. The (standard) nth-order prolongation of $\mathbf{X}$ is the vector field

$$\mathbf{X}^{(n)} = \mathbf{X} + \sum_{i=1}^{n} \eta^{(i)}(t, x^{(i)})\partial_{x_i},$$

defined on $M^{(n)}$, where

$$\eta^{(i)}(t, x^{(i)}) = D_x(\eta^{(i-1)}(t, x^{(i-1)})) - x_i D_t(\xi(t,x)), \quad i = 1, \ldots, n.$$

P. J. Olver 1986.
Applications of Lie groups to differential equations.
Springer, New York.

Definition

A smooth vector field $\mathbf{X} = \xi(t,x)\partial_t + \eta(t,x)\partial_x$ defined on M is a Lie point symmetry of equation (1) if and only if

1.
$$\mathbf{X}^{(n)}(x_n - F(t,x,\ldots,x_{n-1})) = 0 \quad \text{if} \quad x_n = F(t,x,\ldots,x_{n-1}), \tag{3}$$

P. J. Olver 1986.
Applications of Lie groups to differential equations.
Springer, New York.

Definition

A smooth vector field $\mathbf{X} = \xi(t,x)\partial_t + \eta(t,x)\partial_x$ defined on M is a Lie point symmetry of equation (1) if and only if

1. $$\mathbf{X}^{(n)}(x_n - F(t,x,\dots,x_{n-1})) = 0 \quad \text{if} \quad x_n = F(t,x,\dots,x_{n-1}), \qquad (3)$$

2. or equivalently
$$[\mathbf{X}^{(n-1)}, \mathbf{A}] = -\mathbf{A}(\xi)\mathbf{A}. \qquad (4)$$

P. J. Olver 1986.
Applications of Lie groups to differential equations.
Springer, New York.

Use of a Lie point symmetry to reduce the order of equation (1)

- In the neighbourhood of a point where $\mathbf{X}$ does not vanish there exist two functions $z = z(t, x)$ and $\alpha = \alpha(t, x)$ such that

$$\mathbf{X}(z) = 0 \quad \text{and} \quad \mathbf{X}(\alpha) = 1.$$

Use of a Lie point symmetry to reduce the order of equation (1)

- In the neighbourhood of a point where $\mathbf{X}$ does not vanish there exist two functions $z = z(t, x)$ and $\alpha = \alpha(t, x)$ such that

$$\mathbf{X}(z) = 0 \quad \text{and} \quad \mathbf{X}(\alpha) = 1.$$

- Consider $w_{i-1} = \frac{d^i \alpha}{dz^i}$, for $i = 1, \ldots, n$, where $w_0 = w$.

Use of a Lie point symmetry to reduce the order of equation (1)

- In the neighbourhood of a point where $\mathbf{X}$ does not vanish there exist two functions $z = z(t, x)$ and $\alpha = \alpha(t, x)$ such that

$$\mathbf{X}(z) = 0 \quad \text{and} \quad \mathbf{X}(\alpha) = 1.$$

- Consider $w_{i-1} = \frac{d^i \alpha}{dz^i}$, for $i = 1, \ldots, n$, where $w_0 = w$.
- Then we have that $\mathbf{X}^{(i)}(w_{i-1}) = 0$, for $i = 1, \ldots, n$.

Use of a Lie point symmetry to reduce the order of equation (1)

- In the neighbourhood of a point where $\mathbf{X}$ does not vanish there exist two functions $z = z(t, x)$ and $\alpha = \alpha(t, x)$ such that

$$\mathbf{X}(z) = 0 \quad \text{and} \quad \mathbf{X}(\alpha) = 1.$$

- Consider $w_{i-1} = \frac{d^i \alpha}{dz^i}$, for $i = 1, \ldots, n$, where $w_0 = w$.
- Then we have that $\mathbf{X}^{(i)}(w_{i-1}) = 0$, for $i = 1, \ldots, n$.
- Consider the local change of variables

$$\varphi(t, x, \ldots, x_n) = (z, \alpha, w, \ldots, w_{n-1}). \tag{5}$$

- Suppose that equation (1) becomes

$$\widetilde{F}(y, \alpha, w, \dots, w_{n-1}) = 0 \tag{6}$$

in the new variables.

- Suppose that equation (1) becomes

$$\widetilde{F}(y, \alpha, w, \dots, w_{n-1}) = 0 \tag{6}$$

  in the new variables.
- The vector field $\mathbf{X}^{(n)}$ in the new variables becomes

$$\widetilde{\mathbf{X}^{(n)}} = \partial_\alpha,$$

- Suppose that equation (1) becomes

$$\widetilde{F}(y, \alpha, w, \dots, w_{n-1}) = 0 \tag{6}$$

in the new variables.

- The vector field $\mathbf{X}^{(n)}$ in the new variables becomes

$$\widetilde{\mathbf{X}^{(n)}} = \partial_\alpha,$$

- therefore the infinitesimal criteria establishes that

$$\frac{\partial \widetilde{F}}{\partial \alpha} = 0 \quad \text{if} \quad \widetilde{F} = 0.$$

- Suppose that equation (1) becomes

$$\widetilde{F}(y, \alpha, w, \dots, w_{n-1}) = 0 \tag{6}$$

  in the new variables.
- The vector field $\mathbf{X}^{(n)}$ in the new variables becomes

$$\widetilde{\mathbf{X}^{(n)}} = \partial_\alpha,$$

- therefore the infinitesimal criteria establishes that

$$\frac{\partial \widetilde{F}}{\partial \alpha} = 0 \quad \text{if} \quad \widetilde{F} = 0.$$

- We obtain a reduced equation

$$\widetilde{F}(z, w, \cdots, w_{n-1}) = 0. \tag{7}$$

- **Remark:** If $n = 1$ then we have that $\widetilde{F}(z, w) = 0$, hence we can locally obtain
$$w = \frac{d\alpha}{dz} = g(z).$$
In this case the solution can be obtained by a single quadrature:
$$\alpha = \int g(z)dz + C, \quad C \in \mathbb{R}.$$

- **Remark:** If $n = 1$ then we have that $\widetilde{F}(z, w) = 0$, hence we can locally obtain
$$w = \frac{d\alpha}{dz} = g(z).$$
In this case the solution can be obtained by a single quadrature:
$$\alpha = \int g(z)dz + C, \quad C \in \mathbb{R}.$$
- **Recovery of solutions**
Suppose that $w = H(z; C_1, \ldots, C_{n-1})$ is the general solution of (7).

- **Remark:** If $n = 1$ then we have that $\widetilde{F}(z, w) = 0$, hence we can locally obtain

$$w = \frac{d\alpha}{dz} = g(z).$$

In this case the solution can be obtained by a single quadrature:

$$\alpha = \int g(z)dz + C, \quad C \in \mathbb{R}.$$

- **Recovery of solutions**
Suppose that $w = H(z; C_1, \ldots, C_{n-1})$ is the general solution of (7).
- Auxiliary equation:

$$w(t, x, x_1) = H(z(t, x); C_1, \ldots, C_{n-1}),$$

- **Remark:** If $n = 1$ then we have that $\widetilde{F}(z, w) = 0$, hence we can locally obtain
$$w = \frac{d\alpha}{dz} = g(z).$$
In this case the solution can be obtained by a single quadrature:
$$\alpha = \int g(z)dz + C, \quad C \in \mathbb{R}.$$
- **Recovery of solutions**
Suppose that $w = H(z; C_1, \ldots, C_{n-1})$ is the general solution of (7).
- Auxiliary equation:
$$w(t, x, x_1) = H(z(t, x); C_1, \ldots, C_{n-1}),$$
- **X** is a Lie symmetry of the auxiliary equation.

- Let $\mathcal{L}$ be the set of Lie symmetries of equation (1).

- Let $\mathcal{L}$ be the set of Lie symmetries of equation (1).
- Then we have that:

- Let $\mathcal{L}$ be the set of Lie symmetries of equation (1).
- Then we have that:
  1. If $\mathbf{X}, \mathbf{Y} \in \mathcal{L}$ and $\alpha, \beta \in \mathbb{R}$ then $\alpha\mathbf{X} + \beta\mathbf{Y} \in \mathcal{L}$.

- Let $\mathcal{L}$ be the set of Lie symmetries of equation (1).
- Then we have that:
  1. If $\mathbf{X}, \mathbf{Y} \in \mathcal{L}$ and $\alpha, \beta \in \mathbb{R}$ then $\alpha\mathbf{X} + \beta\mathbf{Y} \in \mathcal{L}$.
  2. If $\mathbf{X}, \mathbf{Y} \in \mathcal{L}$ then $[\mathbf{X}, \mathbf{Y}] \in \mathcal{L}$.

- Let $\mathcal{L}$ be the set of Lie symmetries of equation (1).
- Then we have that:
  1. If $\mathbf{X}, \mathbf{Y} \in \mathcal{L}$ and $\alpha, \beta \in \mathbb{R}$ then $\alpha\mathbf{X} + \beta\mathbf{Y} \in \mathcal{L}$.
  2. If $\mathbf{X}, \mathbf{Y} \in \mathcal{L}$ then $[\mathbf{X}, \mathbf{Y}] \in \mathcal{L}$.
- Therefore $\mathcal{L}$ has the structure of real Lie algebra with respect to the usual Lie bracket of vector fields.

- Let $\mathcal{L}$ be the set of Lie symmetries of equation (1).
- Then we have that:
  1. If $\mathbf{X}, \mathbf{Y} \in \mathcal{L}$ and $\alpha, \beta \in \mathbb{R}$ then $\alpha\mathbf{X} + \beta\mathbf{Y} \in \mathcal{L}$.
  2. If $\mathbf{X}, \mathbf{Y} \in \mathcal{L}$ then $[\mathbf{X}, \mathbf{Y}] \in \mathcal{L}$.
- Therefore $\mathcal{L}$ has the structure of real Lie algebra with respect to the usual Lie bracket of vector fields.
- $\mathcal{L}$ is called the Lie symmetry algebra of equation (1).

Use of a Lie symmetry algebra to reduce the order of equation (1)

- Assume that $\mathcal{L} \neq \emptyset$ and $\dim(\mathcal{L}) = k$, $2 \leq k \leq n$.

Use of a Lie symmetry algebra to reduce the order of equation (1)

- Assume that $\mathcal{L} \neq \emptyset$ and $\dim(\mathcal{L}) = k$, $2 \leq k \leq n$.
- By using $\mathbf{X}_1 \in \mathcal{L}$ we can reduce the order of equation (1) by one:

$$\widetilde{F}(z, w, \ldots, w_{n-1}) = 0.$$

Use of a Lie symmetry algebra to reduce the order of equation (1)

- Assume that $\mathcal{L} \neq \emptyset$ and $\dim(\mathcal{L}) = k$, $2 \leq k \leq n$.
- By using $\mathbf{X}_1 \in \mathcal{L}$ we can reduce the order of equation (1) by one:

$$\widetilde{F}(z, w, \dots, w_{n-1}) = 0.$$

- We would like to use another Lie symmetry $\mathbf{X}_2 \in \mathcal{L}$, $\mathbf{X}_2 \neq \mathbf{X}_1$, to reduce the order again.

Use of a Lie symmetry algebra to reduce the order of equation (1)

- Assume that $\mathcal{L} \neq \emptyset$ and $\dim(\mathcal{L}) = k$, $2 \leq k \leq n$.
- By using $\mathbf{X}_1 \in \mathcal{L}$ we can reduce the order of equation (1) by one:

$$\widetilde{F}(z, w, \dots, w_{n-1}) = 0.$$

- We would like to use another Lie symmetry $\mathbf{X}_2 \in \mathcal{L}$, $\mathbf{X}_2 \neq \mathbf{X}_1$, to reduce the order again.
- The vector field $\mathbf{X}_2^{(1)}$ becomes in terms of the coordinates (z, α, w):

$$\widetilde{\mathbf{X}_2^{(1)}} = \mathbf{X}_2^{(1)}(z)\partial_z + \mathbf{X}_2^{(1)}(\alpha)\partial_\alpha + \mathbf{X}_2^{(1)}(w)\partial_w.$$

Use of a Lie symmetry algebra to reduce the order of equation (1)

- Assume that $\mathcal{L} \neq \emptyset$ and $\dim(\mathcal{L}) = k$, $2 \leq k \leq n$.
- By using $\mathbf{X}_1 \in \mathcal{L}$ we can reduce the order of equation (1) by one:

$$\widetilde{F}(z, w, \dots, w_{n-1}) = 0.$$

- We would like to use another Lie symmetry $\mathbf{X}_2 \in \mathcal{L}$, $\mathbf{X}_2 \neq \mathbf{X}_1$, to reduce the order again.
- The vector field $\mathbf{X}_2^{(1)}$ becomes in terms of the coordinates (z, α, w):

$$\widetilde{\mathbf{X}_2^{(1)}} = \mathbf{X}_2^{(1)}(z)\partial_z + \mathbf{X}_2^{(1)}(\alpha)\partial_\alpha + \mathbf{X}_2^{(1)}(w)\partial_w.$$

- We have to project $\widetilde{\mathbf{X}_2^{(1)}}$ to the space of coordinates (y, w).

- For that

$$\frac{\partial \mathbf{X}_2^{(1)}(z)}{\partial \alpha} = 0 \quad \text{and} \quad \frac{\partial \mathbf{X}_2^{(1)}(w)}{\partial \alpha} = 0,$$

- For that

$$\frac{\partial \mathbf{X}_2^{(1)}(z)}{\partial \alpha} = 0 \quad \text{and} \quad \frac{\partial \mathbf{X}_2^{(1)}(w)}{\partial \alpha} = 0,$$

- which happen if and only if

$$[\mathbf{X}_1, \mathbf{X}_2] = c\mathbf{X}_1, \quad c \in \mathbb{R}.$$

- For that

$$\frac{\partial \mathbf{X}_2^{(1)}(z)}{\partial \alpha} = 0 \quad \text{and} \quad \frac{\partial \mathbf{X}_2^{(1)}(w)}{\partial \alpha} = 0,$$

- which happen if and only if

$$[\mathbf{X}_1, \mathbf{X}_2] = c\mathbf{X}_1, \quad c \in \mathbb{R}.$$

- Let us assume that $\mathcal{L}$ is solvable, i.e, it admits a decomposition of the form

$$\langle \mathbf{X}_1 \rangle \triangleleft \langle \mathbf{X}_1, \mathbf{X}_2 \rangle \triangleleft \cdots \triangleleft \langle \mathbf{X}_1, \ldots, \mathbf{X}_k \rangle.$$

- For that

$$\frac{\partial \mathbf{X}_2^{(1)}(z)}{\partial \alpha} = 0 \quad \text{and} \quad \frac{\partial \mathbf{X}_2^{(1)}(w)}{\partial \alpha} = 0,$$

- which happen if and only if

$$[\mathbf{X}_1, \mathbf{X}_2] = c\mathbf{X}_1, \quad c \in \mathbb{R}.$$

- Let us assume that $\mathcal{L}$ is solvable, i.e, it admits a decomposition of the form

$$\langle \mathbf{X}_1 \rangle \triangleleft \langle \mathbf{X}_1, \mathbf{X}_2 \rangle \triangleleft \cdots \triangleleft \langle \mathbf{X}_1, \ldots, \mathbf{X}_k \rangle.$$

- Then the order of the equation (1) can be stepwise reduced by k.

λ-symmetries

Definition (λ-prolongation of vector fields)

For a given smooth vector field $\mathbf{X} = \xi(t,x)\partial_t + \eta(t,x)\partial_x$ defined on M and for an arbitrary function $\lambda \in \mathcal{C}^\infty(M^{(1)})$, the *$\lambda$-prolongation of order n* of $\mathbf{X}$ is the vector field

$$\mathbf{X}^{[\lambda,(n)]} = \xi(t,x)\partial_t + \sum_{i=0}^{n} \eta^{[\lambda,(i)]}(t,x^{(i)})\partial_{x_i}, \tag{8}$$

defined on $M^{(n)}$, where $\eta^{[\lambda,(0)]}(t,x) = \eta(t,x)$ and, for $1 \leq i \leq n$,

$$\begin{aligned} \eta^{[\lambda,(i)]}(t,x^{(i)}) \quad = \quad & D_t\left(\eta^{[\lambda,(i-1)]}(t,x^{(i-1)})\right) - D_t(\xi(t,x))x_i \\ & +\lambda\left(\eta^{[\lambda,(i-1)]}(t,x^{(i-1)}) - \xi(t,x)x_i\right). \end{aligned} \tag{9}$$

Muriel, C. and Romero, J. L. 2001.
New methods of reduction for ordinary differential equations.
IMA Journal of Applied Mathematics **66** 111-125

Definition

A pair $(\mathbf{X}, \lambda)$, where $\mathbf{X} = \xi(t,x)\partial_t + \eta(t,x)\partial_x$ is a smooth vector field defined on M and $\lambda \in \mathcal{C}^\infty(M^{(1)})$, is a λ-symmetry of equation (1) if and only if

1. $$\mathbf{X}^{[\lambda,(n)]}(x_n - F(t,x,\ldots,x_{n-1})) = 0 \quad \text{if} \;\; x_n = F(t,x,\ldots,x_{n-1}), \quad (10)$$

Muriel, C. and Romero, J. L. 2001.
New methods of reduction for ordinary differential equations.
IMA Journal of Applied Mathematics **66** 111-125

Definition

A pair $(\mathbf{X}, \lambda)$, where $\mathbf{X} = \xi(t,x)\partial_t + \eta(t,x)\partial_x$ is a smooth vector field defined on M and $\lambda \in \mathcal{C}^\infty(M^{(1)})$, is a λ-symmetry of equation (1) if and only if

1. $$\mathbf{X}^{[\lambda,(n)]}(x_n - F(t,x,\dots,x_{n-1})) = 0 \quad \text{if} \quad x_n = F(t,x,\dots,x_{n-1}), \qquad (10)$$

2. or equivalently

$$[\mathbf{X}^{[\lambda,(n-1)]}, \mathbf{A}] = -(\mathbf{A}(\xi) + \lambda\xi)\mathbf{A} + \lambda\mathbf{X}^{[\lambda,(n-1)]}. \qquad (11)$$

Muriel, C. and Romero, J. L. 2001.
New methods of reduction for ordinary differential equations.
IMA Journal of Applied Mathematics **66** 111-125

Use of λ-symmetries to reduce the order of equation (1)

- Suppose that $(\mathbf{X}, \lambda)$ is a λ-symmetry of equation (1).

Use of λ-symmetries to reduce the order of equation (1)

- Suppose that $(\mathbf{X}, \lambda)$ is a λ-symmetry of equation (1).
- Let $y = y(t, x)$ and $w = w(t, x, x_1)$ be two functions such that

$$\mathbf{X}^{[\lambda,(1)]}(y) = \mathbf{X}^{[\lambda,(1)]}(w) = 0.$$

Use of λ-symmetries to reduce the order of equation (1)

- Suppose that $(\mathbf{X}, \lambda)$ is a λ-symmetry of equation (1).
- Let $y = y(t, x)$ and $w = w(t, x, x_1)$ be two functions such that

$$\mathbf{X}^{[\lambda,(1)]}(y) = \mathbf{X}^{[\lambda,(1)]}(w) = 0.$$

- (IBDP) If we consider the invariants obtained by derivation

$$w_i = \frac{d^{(i-1)} w}{dy^{(i-1)}}, \quad i = 1, \ldots, n-1,$$

then we have that

$$\mathbf{X}^{[\lambda,(i)]}(w_i) = 0, \quad i = 1, \ldots, n-1.$$

Use of λ-symmetries to reduce the order of equation (1)

- Suppose that $(\mathbf{X}, \lambda)$ is a λ-symmetry of equation (1).
- Let $y = y(t, x)$ and $w = w(t, x, x_1)$ be two functions such that

$$\mathbf{X}^{[\lambda,(1)]}(y) = \mathbf{X}^{[\lambda,(1)]}(w) = 0.$$

- (IBDP) If we consider the invariants obtained by derivation

$$w_i = \frac{d^{(i-1)}w}{dy^{(i-1)}}, \quad i = 1, \dots, n-1,$$

then we have that

$$\mathbf{X}^{[\lambda,(i)]}(w_i) = 0, \quad i = 1, \dots, n-1.$$

- Consider the local change of variables

$$\phi(t, x, x_1, \dots, x_n) = (y, \beta, w, \dots, w_{n-1}),$$

where β is some function such that $\mathbf{X}(\beta) \neq 0$.

- By the invariance criteria we obtain a reduced equation of the form

$$\widetilde{F}(y, w, \ldots, w_{n-1}) = 0.$$

- By the invariance criteria we obtain a reduced equation of the form

$$\widetilde{F}(y, w, \ldots, w_{n-1}) = 0.$$

- Suppose that $w = H(y; C_1, \ldots, C_{n-1})$ is the general solution of the above reduced equation.

- By the invariance criteria we obtain a reduced equation of the form

$$\widetilde{F}(y, w, \ldots, w_{n-1}) = 0.$$

- Suppose that $w = H(y; C_1, \ldots, C_{n-1})$ is the general solution of the above reduced equation.
- Auxiliary equation:

$$w(t, x, x_1) = H(y(t, x); C_1, \ldots, C_{n-1}).$$

Liénard I-type equation

- The Liénard I-type equation

$$x_2 + a_1(x)x_1 + a_0(x) = 0, \tag{12}$$

where a_1 and a_0 are arbitrary smooth functions of the dependent variable x and $x_i = \frac{d^i x}{dt^i}$ for $i = 1, 2$.

Liénard I-type equation

- The Liénard I-type equation

$$x_2 + a_1(x)x_1 + a_0(x) = 0, \qquad (12)$$

where a_1 and a_0 are arbitrary smooth functions of the dependent variable x and $x_i = \frac{d^i x}{dt^i}$ for $i = 1, 2$.

- $\mathbf{A} = \partial_t + x_1\partial_x - (a_1(x)x_1 + a_0(x))\partial_{x_1}$.

Liénard I-type equation

- The Liénard I-type equation

$$x_2 + a_1(x)x_1 + a_0(x) = 0, \qquad (12)$$

  where a_1 and a_0 are arbitrary smooth functions of the dependent variable x and $x_i = \frac{d^i x}{dt^i}$ for $i = 1, 2$.
- $\mathbf{A} = \partial_t + x_1\partial_x - (a_1(x)x_1 + a_0(x))\partial_{x_1}$.
- Equation (12) models some famous nonlinear oscillators such as the van der Pol equation, the Duffing oscillator, the Helmholtz oscillator, etc.

Liénard I-type equation

- The Liénard I-type equation

$$x_2 + a_1(x)x_1 + a_0(x) = 0, \tag{12}$$

  where a_1 and a_0 are arbitrary smooth functions of the dependent variable x and $x_i = \frac{d^i x}{dt^i}$ for $i = 1, 2$.
- $\mathbf{A} = \partial_t + x_1\partial_x - (a_1(x)x_1 + a_0(x))\partial_{x_1}$.
- Equation (12) models some famous nonlinear oscillators such as the van der Pol equation, the Duffing oscillator, the Helmholtz oscillator, etc.
- It appears as reductions of nonlinear partial differential equations (PDEs) such as the Fisher equation, the Burgers-Korteweg-de Vries equation, and the Burgers-Huxley equation.

- We have that $\mathbf{X}_1 = \partial_t$ is a Lie point symmetry of equation (12).

- We have that $\mathbf{X}_1 = \partial_t$ is a Lie point symmetry of equation (12).
- The associated reduced equation by means of the transformation $w = 1/x_1$ becomes

$$w'(x) = a_1(x)w(x)^2 + a_0(x)w(x)^3. \tag{13}$$

- We have that $\mathbf{X}_1 = \partial_t$ is a Lie point symmetry of equation (12).
- The associated reduced equation by means of the transformation $w = 1/x_1$ becomes

$$w'(x) = a_1(x)w(x)^2 + a_0(x)w(x)^3. \tag{13}$$

- Equation (13) is an Abel equation of the first kind and its integrability by quadratures cannot be guaranteed in general

- We have that $\mathbf{X}_1 = \partial_t$ is a Lie point symmetry of equation (12).
- The associated reduced equation by means of the transformation $w = 1/x_1$ becomes

$$w'(x) = a_1(x)w(x)^2 + a_0(x)w(x)^3. \tag{13}$$

- Equation (13) is an Abel equation of the first kind and its integrability by quadratures cannot be guaranteed in general
- If $J_1 = J_1(t, w)$ denotes a first integral of (13) then the function J_1 written in terms of the original variables

$$I_1(x, x_1) = J_1\left(x, \frac{1}{x_1}\right) \tag{14}$$

is a common first integral of the system of vector fields $\{\mathbf{A}, \mathbf{X}_1\}$.

- We need to obtain an explicit solution $w = H(t, K_1)$ of equation (13) from $J_1(t, w) = K_1$, where $K_1 \in \mathbb{R}$.

- We need to obtain an explicit solution $w = H(t, K_1)$ of equation (13) from $J_1(t, w) = K_1$, where $K_1 \in \mathbb{R}$.
- The general solution of equation (12) can be obtained after evaluating the quadrature

$$\int \frac{dx}{H(x, K_1)} = t + K_2, \quad K_2 \in \mathbb{R}. \tag{15}$$

- We need to obtain an explicit solution $w = H(t, K_1)$ of equation (13) from $J_1(t, w) = K_1$, where $K_1 \in \mathbb{R}$.
- The general solution of equation (12) can be obtained after evaluating the quadrature

$$\int \frac{dx}{H(x, K_1)} = t + K_2, \quad K_2 \in \mathbb{R}. \tag{15}$$

- We look for λ-symmetries to equation (12).

λ-symmetries of Liénard I-type equations

- A pair (∂_x, λ) is a λ-symmetry of the Liénard I-type equation if and only if

$$\lambda_t + \lambda_x x_1 - \lambda_{x_1}\left(a_1 x_1 + a_0\right) + \lambda^2 = -a_1' x_1 - a_0' - a_1 \lambda. \tag{16}$$

λ-symmetries of Liénard I-type equations

- A pair (∂_x, λ) is a λ-symmetry of the Liénard I-type equation if and only if

$$\lambda_t + \lambda_x x_1 - \lambda_{x_1}\left(a_1 x_1 + a_0\right) + \lambda^2 = -a_1' x_1 - a_0' - a_1 \lambda. \qquad (16)$$

- For second order ordinary differential equations, a first integral $I_2 = I_2(t, x, x_1)$ is always associated to a λ-symmetry of the equation of the form $\left(\partial_x, \lambda\right)$.

λ-symmetries of Liénard I-type equations

- A pair (∂_x, λ) is a λ-symmetry of the Liénard I-type equation if and only if

$$\lambda_t + \lambda_x x_1 - \lambda_{x_1}\left(a_1 x_1 + a_0\right) + \lambda^2 = -a_1' x_1 - a_0' - a_1 \lambda. \qquad (16)$$

- For second order ordinary differential equations, a first integral $I_2 = I_2(t, x, x_1)$ is always associated to a λ-symmetry of the equation of the form $\left(\partial_x, \lambda\right)$.
- The first integral I_1, given in (14), and the first integral I_2, are functionally independent if and only if

$$\lambda \neq \mathbf{A}(Q_1)/Q_1 = -\frac{a_1 x_1 + a_0}{x_1},$$

where $Q_1 = -x_1$ is the characteristic of the vector field $\mathbf{X}_1 = \partial_t$.

How can the λ-symmetry be used to compute I_2 by quadratures?

- We construct a solvable structure.

How can the λ-symmetry be used to compute I_2 by quadratures?

- We construct a solvable structure.
- Consider $\mathbf{X}_2 = \partial_x + \lambda \partial_{x_1}$.

How can the λ-symmetry be used to compute I_2 by quadratures?

- We construct a solvable structure.
- Consider $\mathbf{X}_2 = \partial_x + \lambda \partial_{x_1}$.
- It can be checked that

$$[\mathbf{X}_2, \mathbf{A}] = \lambda \mathbf{X}_2, \tag{17}$$

$$[\mathbf{X}_1, \mathbf{A}] = 0 \quad \text{and} \quad [\mathbf{X}_1, \mathbf{X}_2] = \frac{\lambda_t}{\lambda x_1 + a_1 x_1 + a_0} \left(\mathbf{X}_1 + \mathbf{X}_2 - \mathbf{A} \right). \tag{18}$$

How can the λ-symmetry be used to compute I_2 by quadratures?

- We construct a solvable structure.
- Consider $\mathbf{X}_2 = \partial_x + \lambda \partial_{x_1}$.
- It can be checked that

$$[\mathbf{X}_2, \mathbf{A}] = \lambda \mathbf{X}_2, \tag{17}$$

$$[\mathbf{X}_1, \mathbf{A}] = 0 \quad \text{and} \quad [\mathbf{X}_1, \mathbf{X}_2] = \frac{\lambda_t}{\lambda x_1 + a_1 x_1 + a_0} \left(\mathbf{X}_1 + \mathbf{X}_2 - \mathbf{A} \right). \tag{18}$$

- The ordered set of vector fields $\langle \mathbf{X}_2, \mathbf{A}, \mathbf{X}_1 \rangle$ is a solvable structure with respect to $\mathbf{X}_2$ if and only if the following three conditions hold:

How can the λ-symmetry be used to compute I_2 by quadratures?

- We construct a solvable structure.
- Consider $\mathbf{X}_2 = \partial_x + \lambda \partial_{x_1}$.
- It can be checked that

$$[\mathbf{X}_2, \mathbf{A}] = \lambda \mathbf{X}_2, \tag{17}$$

$$[\mathbf{X}_1, \mathbf{A}] = 0 \quad \text{and} \quad [\mathbf{X}_1, \mathbf{X}_2] = \frac{\lambda_t}{\lambda x_1 + a_1 x_1 + a_0} \left(\mathbf{X}_1 + \mathbf{X}_2 - \mathbf{A} \right). \tag{18}$$

- The ordered set of vector fields $\langle \mathbf{X}_2, \mathbf{A}, \mathbf{X}_1 \rangle$ is a solvable structure with respect to $\mathbf{X}_2$ if and only if the following three conditions hold:
  (a) $[\mathbf{X}_2, \mathbf{A}] \in \text{span}\{\mathbf{X}_2\}$,

How can the λ-symmetry be used to compute I_2 by quadratures?

- We construct a solvable structure.
- Consider $\mathbf{X}_2 = \partial_x + \lambda\partial_{x_1}$.
- It can be checked that

$$[\mathbf{X}_2, \mathbf{A}] = \lambda \mathbf{X}_2, \tag{17}$$

$$[\mathbf{X}_1, \mathbf{A}] = 0 \quad \text{and} \quad [\mathbf{X}_1, \mathbf{X}_2] = \frac{\lambda_t}{\lambda x_1 + a_1 x_1 + a_0} \left(\mathbf{X}_1 + \mathbf{X}_2 - \mathbf{A} \right). \tag{18}$$

- The ordered set of vector fields $\langle \mathbf{X}_2, \mathbf{A}, \mathbf{X}_1 \rangle$ is a solvable structure with respect to $\mathbf{X}_2$ if and only if the following three conditions hold:
  - (a) $[\mathbf{X}_2, \mathbf{A}] \in \text{span}\{\mathbf{X}_2\}$,
  - (b) $[\mathbf{A}, \mathbf{X}_1] \in \text{span}\{\mathbf{X}_2, \mathbf{A}\}$,

How can the λ-symmetry be used to compute I_2 by quadratures?

- We construct a solvable structure.
- Consider $\mathbf{X}_2 = \partial_x + \lambda \partial_{x_1}$.
- It can be checked that

$$[\mathbf{X}_2, \mathbf{A}] = \lambda \mathbf{X}_2, \quad (17)$$

$$[\mathbf{X}_1, \mathbf{A}] = 0 \quad \text{and} \quad [\mathbf{X}_1, \mathbf{X}_2] = \frac{\lambda_t}{\lambda x_1 + a_1 x_1 + a_0} \left(\mathbf{X}_1 + \mathbf{X}_2 - \mathbf{A} \right). \quad (18)$$

- The ordered set of vector fields $\langle \mathbf{X}_2, \mathbf{A}, \mathbf{X}_1 \rangle$ is a solvable structure with respect to $\mathbf{X}_2$ if and only if the following three conditions hold:
  - (a) $[\mathbf{X}_2, \mathbf{A}] \in \text{span}\{\mathbf{X}_2\}$,
  - (b) $[\mathbf{A}, \mathbf{X}_1] \in \text{span}\{\mathbf{X}_2, \mathbf{A}\}$,
  - (c) $[\mathbf{X}_2, \mathbf{X}_1] \in \text{span}\{\mathbf{X}_2, \mathbf{A}\}$.

How can the λ-symmetry be used to compute I_2 by quadratures?

- We construct a solvable structure.
- Consider $\mathbf{X}_2 = \partial_x + \lambda \partial_{x_1}$.
- It can be checked that

$$[\mathbf{X}_2, \mathbf{A}] = \lambda \mathbf{X}_2, \tag{17}$$

$$[\mathbf{X}_1, \mathbf{A}] = 0 \quad \text{and} \quad [\mathbf{X}_1, \mathbf{X}_2] = \frac{\lambda_t}{\lambda x_1 + a_1 x_1 + a_0} \left(\mathbf{X}_1 + \mathbf{X}_2 - \mathbf{A} \right). \tag{18}$$

- The ordered set of vector fields $\langle \mathbf{X}_2, \mathbf{A}, \mathbf{X}_1 \rangle$ is a solvable structure with respect to $\mathbf{X}_2$ if and only if the following three conditions hold:
  (a) $[\mathbf{X}_2, \mathbf{A}] \in \text{span}\{\mathbf{X}_2\}$,
  (b) $[\mathbf{A}, \mathbf{X}_1] \in \text{span}\{\mathbf{X}_2, \mathbf{A}\}$,
  (c) $[\mathbf{X}_2, \mathbf{X}_1] \in \text{span}\{\mathbf{X}_2, \mathbf{A}\}$.

Theorem

Then the ordered set of vector fields $\langle \mathbf{X}_2, \mathbf{A}, \mathbf{X}_1 \rangle$ is a solvable structure with respect to $\mathbf{X}_2$ if and only if $\lambda_t = 0$.

- Let $\mathbf{\Omega} = dt \wedge dx \wedge dx_1$ be the volume form on $M^{(1)}$.

- Let $\mathbf{\Omega} = dt \wedge dx \wedge dx_1$ be the volume form on $M^{(1)}$.
- If $\langle \mathbf{X}_2, \mathbf{A}, \mathbf{X}_1 \rangle$ is a solvable structure with respect to $\mathbf{X}_2$ then the differential 1-form

$$\boldsymbol{\omega}_2 = \frac{\mathbf{A} \lrcorner \mathbf{X}_2 \lrcorner \mathbf{\Omega}}{\mathbf{X}_1 \lrcorner \mathbf{A} \lrcorner \mathbf{X}_2 \lrcorner \mathbf{\Omega}} \tag{19}$$

is locally exact, and a function I_2 such that $dI_2 = \boldsymbol{\omega}_2$ is a common first integral of the system of vector fields $\{\mathbf{A}, \mathbf{X}_2\}$.

- Let $\mathbf{\Omega} = dt \wedge dx \wedge dx_1$ be the volume form on $M^{(1)}$.
- If $\langle \mathbf{X}_2, \mathbf{A}, \mathbf{X}_1 \rangle$ is a solvable structure with respect to $\mathbf{X}_2$ then the differential 1-form

$$\boldsymbol{\omega}_2 = \frac{\mathbf{A} \lrcorner \mathbf{X}_2 \lrcorner \mathbf{\Omega}}{\mathbf{X}_1 \lrcorner \mathbf{A} \lrcorner \mathbf{X}_2 \lrcorner \mathbf{\Omega}} \tag{19}$$

is locally exact, and a function I_2 such that $dI_2 = \boldsymbol{\omega}_2$ is a common first integral of the system of vector fields $\{\mathbf{A}, \mathbf{X}_2\}$.

J. Sherring, G. Prince 1992.
Geometric aspects of reduction of order.
Transactions of the American Mathematical Society **334** 433–453

- Let $\mathbf{\Omega} = dt \wedge dx \wedge dx_1$ be the volume form on $M^{(1)}$.
- If $\langle \mathbf{X}_2, \mathbf{A}, \mathbf{X}_1 \rangle$ is a solvable structure with respect to $\mathbf{X}_2$ then the differential 1-form

$$\omega_2 = \frac{\mathbf{A} \lrcorner \mathbf{X}_2 \lrcorner \mathbf{\Omega}}{\mathbf{X}_1 \lrcorner \mathbf{A} \lrcorner \mathbf{X}_2 \lrcorner \mathbf{\Omega}} \tag{19}$$

  is locally exact, and a function I_2 such that $dI_2 = \omega_2$ is a common first integral of the system of vector fields $\{\mathbf{A}, \mathbf{X}_2\}$.
- $$\omega_2 = dt - \frac{\lambda}{(\lambda + a_1)x_1 + a_0} dx + \frac{1}{(\lambda + a_1)x_1 + a_0} dx_1.$$
- A primitive $I_2 = I_2(t, x, x_1)$ of ω_2 verifies

$$\frac{\partial I_2}{\partial t} = 1, \quad \frac{\partial I_2}{\partial x} = -\frac{\lambda}{(\lambda + a_1)x_1 + a_0}, \quad \frac{\partial I_2}{\partial x_1} = \frac{1}{(\lambda + a_1)x_1 + a_0}. \tag{20}$$

Theorem

Let $\lambda = \lambda(x, x_1)$ be a function verifying

$$\lambda_x x_1 - \lambda_{x_1}(a_1 x_1 + a_0) + \lambda^2 = -a_1' x_1 - a_0' - a_1 \lambda$$

and such that $\lambda \neq -\dfrac{a_1 x_1 + a_0}{x_1}$. *Then we have that* $I_2(t, x, x_1) = t + F(x, x_1)$, *where*

$$F_x = -\frac{\lambda}{(\lambda + a_1)x_1 + a_0} \quad \text{and} \quad F_{x_1} = \frac{1}{(\lambda + a_1)x_1 + a_0},$$

is a common first integral to the system of vector fields $\{\mathbf{X}_2, \mathbf{A}\}$ *functionally independent to* I_1.

A. Ruiz, C. Muriel 2018.

On the integrability of Liénard I-type equations via λ-symmetries and solvable structures.

Journal of Applied Mathematics and Computation **339** 888–898

Example

- $$x_2 + 4x^2 x_1 + (x^4 + 1)x = 0. \tag{21}$$

Example

- $$x_2 + 4x^2 x_1 + (x^4 + 1)x = 0. \tag{21}$$
- A case of a generalized force-free Duffing-Van der Pol equation.

Example

- $$x_2 + 4x^2 x_1 + (x^4 + 1)x = 0. \tag{21}$$
- A case of a generalized force-free Duffing-Van der Pol equation.
- Equation (21) only admits $\mathbf{X}_1 = \partial_t$ as Lie point symmetry.

Example

- $$x_2 + 4x^2 x_1 + (x^4 + 1)x = 0. \tag{21}$$
- A case of a generalized force-free Duffing-Van der Pol equation.
- Equation (21) only admits $\mathbf{X}_1 = \partial_t$ as Lie point symmetry.
- Reduced Abel equation:

$$w_1 = 4x^2 w^2 + x(x^4 + 1)w^3.$$

Example

- $$x_2 + 4x^2 x_1 + (x^4 + 1)x = 0. \qquad (21)$$
- A case of a generalized force-free Duffing-Van der Pol equation.
- Equation (21) only admits $\mathbf{X}_1 = \partial_t$ as Lie point symmetry.
- Reduced Abel equation:

$$w_1 = 4x^2 w^2 + x(x^4 + 1)w^3.$$

- First integral of the Abel equation:

$$J_1(x, w) = \frac{w\left(\, x + w(1 + x^4)\right)}{x^2 w^2 + (1 + x^3 w)^2} + \arctan\left(\frac{1 + x^3 w}{xw} \right) . \qquad (22)$$

- (∂_x, λ) is a λ-symmetry of (21) for the function

$$\lambda = \frac{x_1}{x} - 2x^2.$$

- (∂_x, λ) is a λ-symmetry of (21) for the function

$$\lambda = \frac{x_1}{x} - 2x^2.$$

- First integral associated to the λ-symmetry:

$$I_2(t, x, x_1) = t + \arctan\left(\frac{x_1 + x^3}{x}\right) . \tag{23}$$

- (∂_x, λ) is a λ-symmetry of (21) for the function

$$\lambda = \frac{x_1}{x} - 2x^2.$$

- First integral associated to the λ-symmetry:

$$I_2(t, x, x_1) = t + \arctan\left(\frac{x_1 + x^3}{x} \right) . \tag{23}$$

- General solution:

$$x(t)^2 = \frac{4\cos^2(K_2 - t)}{\widetilde{K_1} - 4(K_2 - t) - 2\sin(2(K_2 - t))}.$$

A remarkable static Euler-Bernoulli beam equation

- A remarkable static beam equation is

$$y_4 = \delta y^{-5/3}, \qquad \delta = \pm 1. \tag{24}$$

A remarkable static Euler-Bernoulli beam equation

- A remarkable static beam equation is

$$y_4 = \delta y^{-5/3}, \qquad \delta = \pm 1. \tag{24}$$

- The Lie symmetry algebra is three-dimensional, isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ and spanned by the vector fields

$$\mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = x^2\partial_x + 3xy\partial_y, \quad \mathbf{X}_3 = x\partial_x + \frac{3}{2}y\partial_y. \tag{25}$$

A remarkable static Euler-Bernoulli beam equation

- A remarkable static beam equation is

$$y_4 = \delta y^{-5/3}, \qquad \delta = \pm 1. \tag{24}$$

- The Lie symmetry algebra is three-dimensional, isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ and spanned by the vector fields

$$\mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = x^2\partial_x + 3xy\partial_y, \quad \mathbf{X}_3 = x\partial_x + \frac{3}{2}y\partial_y. \tag{25}$$

A.H. Bokhari, F.M. Mahomed, F.D. Zaman 2010.
Symmetries and integrability of a fourth-order Euler-Bernoulli beam equation.
J. Math. Phys. **51** 053517–053526

A remarkable static Euler-Bernoulli beam equation

- A remarkable static beam equation is

$$y_4 = \delta y^{-5/3}, \qquad \delta = \pm 1. \tag{24}$$

- The Lie symmetry algebra is three-dimensional, isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and spanned by the vector fields

$$\mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = x^2\partial_x + 3xy\partial_y, \quad \mathbf{X}_3 = x\partial_x + \frac{3}{2}y\partial_y. \tag{25}$$

A.H. Bokhari, F.M. Mahomed, F.D. Zaman 2010.
Symmetries and integrability of a fourth-order Euler-Bernoulli beam equation.
J. Math. Phys. **51** 053517–053526

- Since the Lie symmetry algebra $\mathfrak{sl}(2,\mathbb{R})$ is nonsolvable, the standard Lie reduction method cannot be used to stepwise reduce the order of (24).

- By means of the transformation $u = y^{2/3}$ the symmetry generators (25) are respectively mapped into

$$\mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = x^2\partial_x + 2xu\partial_u, \quad \mathbf{X}_3 = x\partial_x + u\partial_u. \tag{26}$$

- By means of the transformation $u = y^{2/3}$ the symmetry generators (25) are respectively mapped into

$$\mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = x^2\partial_x + 2xu\partial_u, \quad \mathbf{X}_3 = x\partial_x + u\partial_u. \tag{26}$$

- The corresponding transformed equation is

$$24u^3u_4 + 48u_3u_1u^2 + 36u_2^2u^2 - 36u_2u_1^2u + 9u_1^4 - 16\,\delta = 0, \qquad \delta = \pm 1. \tag{27}$$

- By means of the transformation $u = y^{2/3}$ the symmetry generators (25) are respectively mapped into

$$\mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = x^2\partial_x + 2xu\partial_u, \quad \mathbf{X}_3 = x\partial_x + u\partial_u. \tag{26}$$

- The corresponding transformed equation is

$$24u^3u_4 + 48u_3u_1u^2 + 36u_2^2u^2 - 36u_2u_1^2u + 9u_1^4 - 16\,\delta = 0, \qquad \delta = \pm 1. \tag{27}$$

- A set of joint invariants $\{s, w\}$ for the involutive system of vector fields $\left\{\mathbf{X}_1^{(3)}, \mathbf{X}_2^{(3)}, \mathbf{X}_3^{(3)}\right\}$ is given by

$$s = u_1^2 - 2uu_2 \quad \text{and} \quad w = u^2u_3. \tag{28}$$

- By means of the transformation $u = y^{2/3}$ the symmetry generators (25) are respectively mapped into

$$\mathbf{X}_1 = \partial_x, \quad \mathbf{X}_2 = x^2\partial_x + 2xu\partial_u, \quad \mathbf{X}_3 = x\partial_x + u\partial_u. \tag{26}$$

- The corresponding transformed equation is

$$24u^3u_4 + 48u_3u_1u^2 + 36u_2^2u^2 - 36u_2u_1^2u + 9u_1^4 - 16\,\delta = 0, \qquad \delta = \pm 1. \tag{27}$$

- A set of joint invariants $\{s, w\}$ for the involutive system of vector fields $\left\{\mathbf{X}_1^{(3)}, \mathbf{X}_2^{(3)}, \mathbf{X}_3^{(3)}\right\}$ is given by

$$s = u_1^2 - 2uu_2 \quad \text{and} \quad w = u^2u_3. \tag{28}$$

- Consider the invariant obtained by derivation

$$w_1 = \frac{dw}{ds} = -\frac{2u_3u_1 + uu_4}{u_3}. \tag{29}$$

- Equation (27) can be expressed in terms of the invariants $\{s, w, w_1\}$ as the following reduced equation:

$$2w_1 w = \frac{3}{8}s^2 - \frac{2}{3}\ \delta. \tag{30}$$

- Equation (27) can be expressed in terms of the invariants $\{s, w, w_1\}$ as the following reduced equation:

$$2w_1 w = \frac{3}{8}s^2 - \frac{2}{3}\,\delta. \tag{30}$$

- This is a separable equation that can be integrated by quadratures and whose solutions satisfy:

$$w(s)^2 = \frac{1}{8}s^3 - \frac{2}{3}\,\delta\, s + K, \qquad K \in \mathbb{R}. \tag{31}$$

- Equation (27) can be expressed in terms of the invariants $\{s, w, w_1\}$ as the following reduced equation:

$$2w_1 w = \frac{3}{8}s^2 - \frac{2}{3}\,\delta. \tag{30}$$

- This is a separable equation that can be integrated by quadratures and whose solutions satisfy:

$$w(s)^2 = \frac{1}{8}s^3 - \frac{2}{3}\,\delta\, s + K, \qquad K \in \mathbb{R}. \tag{31}$$

- By isolating K in (31) and by writing the resulting expression in terms of the coordinates $\{x, u, u_1, u_2, u_3\}$, we obtain the following first integral of the fourth-order equation (27):

$$I_0 = u^4 u_3^2 - \frac{1}{8}\left(u_1^2 - 2uu_2\right)^3 + \frac{2}{3}\delta\,\left(u_1^2 - 2uu_2\right). \tag{32}$$

- The reconstruction of the general solution of equation (27) can be carried out by solving the third-order ODEs $I_0 = K_0, K_0 \in \mathbb{R}$, i.e:

$$u^4 u_3^2 - \frac{1}{8}\left(u_1^2 - 2uu_2\right)^3 + \frac{2}{3}\delta\left(u_1^2 - 2uu_2\right) = K_0, \qquad K_0 \in \mathbb{R}. \quad (33)$$

- The reconstruction of the general solution of equation (27) can be carried out by solving the third-order ODEs $I_0 = K_0, K_0 \in \mathbb{R}$, i.e:

$$u^4 u_3^2 - \frac{1}{8}\left(u_1^2 - 2uu_2\right)^3 + \frac{2}{3}\delta\left(u_1^2 - 2uu_2\right) = K_0, \qquad K_0 \in \mathbb{R}. \quad (33)$$

- The above equation inherits $\mathfrak{sl}(2,\mathbb{R})$ as Lie symmetry algebra.

- The reconstruction of the general solution of equation (27) can be carried out by solving the third-order ODEs $I_0 = K_0, K_0 \in \mathbb{R}$, i.e:

$$u^4 u_3^2 - \frac{1}{8}\left(u_1^2 - 2uu_2\right)^3 + \frac{2}{3}\,\delta\,\left(u_1^2 - 2uu_2\right) = K_0, \qquad K_0 \in \mathbb{R}. \quad (33)$$

- The above equation inherits $\mathfrak{sl}(2,\mathbb{R})$ as Lie symmetry algebra.
- The family of third-order ODEs (33) can be locally written as the canonical $\mathrm{SL}(2,\mathbb{R})$-invariant third-order ODE:

$$u_3 = -\frac{1}{8u^2 C(s;K_0)} \qquad (s = u_1^2 - 2uu_2), \quad (34)$$

where the function $C = C(s;K_0)$ satisfies

$$C(s;K_0)^2 = \frac{1}{8\left(s^3 - \frac{16}{3}\,\delta\, s + 8K_0\right)}. \quad (35)$$

Three functionally independent first integrals to the family of third-order ODEs (34) are:

$$I_1 = \frac{2u_1 C(s;K_0)\psi_1(s;K_0) + \psi_1'(s;K_0)}{2u_1 C(s;K_0)\psi_2(s;K_0) + \psi_2'(s;K_0)}, \quad I_2 = \frac{2(u_1 x - 2u)C(s;K_0)\psi_1(s;K_0) + x\psi_1'(s;K_0)}{2(u_1 x - 2u)C(s;K_0)\psi_2(s;K_0) + x\psi_2'(s;K_0)},$$

$$I_3 = \frac{\left(C(s;K_0)2(u_1 x - 2u)\psi_2(s;K_0) + x\psi_2'(s;K_0)\right)^2}{4\,C(s;K_0)uW(\psi_1,\psi_2)(s;K_0)},$$

where $s = u_1^2 - 2uu_2$ and ψ_1 and ψ_2 are two linearly independent solutions to

$$\left(s^3 - \frac{16}{3}\,\delta\,s + 8K_0\right)\psi''(s) + \frac{1}{2}\left(3s^2 - \frac{16}{3}\delta\right)\psi'(s) - \frac{1}{2}s\,\psi(s) = 0. \quad (36)$$

A. Ruiz, C. Muriel 2017.
First Integrals and Parametric Solutions of Third Order ODEs Admitting $\mathfrak{sl}(2,\mathbb{R})$.
J. Phys. A: Math. Theor. **50** 205201

Theorem

A complete set of first integrals $\{I_0, I_1, I_2, I_3\}$ to equation (27) is given by

$$\begin{aligned}
I_0(x,u,u_1,u_2,u_3) &= u^4u_3^2 - \frac{1}{8}\left(u_1^2 - 2uu_2\right)^3 + \frac{2}{3}\,\delta\,\left(u_1^2 - 2uu_2\right),\\
I_1(x,u,u_1,u_2,u_3) &= \frac{u_1\psi_1(s;I_0) - 4u^2u_3\psi_1'(s;I_0)}{u_1\psi_2(s;I_0) - 4u^2u_3\psi_2'(s;I_0)},\\
I_2(x,u,u_1,u_2,u_3) &= \frac{(u_1x-2u)\psi_1(s;I_0) - 4xu^2u_3\psi_1'(s;I_0)}{(u_1x-2u)\psi_2(s;I_0) - 4xu^2u_3\psi_2'(s;I_0)},\\
I_3(x,u,u_1,u_2,u_3) &= \frac{1}{u}\left((u_1x-2u)\psi_2(s;I_0) - 4xu^2u_3\psi_2'(s;I_0)\right)^2,
\end{aligned} \tag{37}$$

where ψ_1 and ψ_2 are two linearly independent solutions to the linear second-order ODE (36) and $s = u_1^2 - 2uu_2$.

- The general solution to the fourth-order ODE (27) is implicitly defined by

$$I_1(x,u,u_1,u_2;K_0)=K_1,\ I_2(x,u,u_1,u_2;K_0)=K_2,\ I_3(x,u,u_1,u_2;K_0)=K_3, \tag{38}$$

where $K_i \in \mathbb{R}$ for $i=0,1,2,3$.

- The general solution to the fourth-order ODE (27) is implicitly defined by

$$I_1(x,u,u_1,u_2;K_0)=K_1,\ I_2(x,u,u_1,u_2;K_0)=K_2,\ I_3(x,u,u_1,u_2;K_0)=K_3, \qquad (38)$$

where $K_i \in \mathbb{R}$ for $i=0,1,2,3$.

- The elimination of u_1 and u_2 from (38) in order to obtain a closed-form solution of equation (27) seems to be impossible, because both functions ψ_1 and ψ_2 and their derivatives are evaluated in $s=u_1^2-2uu_2$.

- The general solution to the fourth-order ODE (27) is implicitly defined by

$$I_1(x, u, u_1, u_2; K_0) = K_1, \; I_2(x, u, u_1, u_2; K_0) = K_2, \; I_3(x, u, u_1, u_2; K_0) = K_3, \qquad (38)$$

 where $K_i \in \mathbb{R}$ for $i = 0, 1, 2, 3$.
- The elimination of u_1 and u_2 from (38) in order to obtain a closed-form solution of equation (27) seems to be impossible, because both functions ψ_1 and ψ_2 and their derivatives are evaluated in $s = u_1^2 - 2uu_2$.
- We focus on obtaining the solution is parametric form.

- The general solution to the fourth-order ODE (27) is implicitly defined by

$$I_1(x,u,u_1,u_2;K_0)=K_1,\ I_2(x,u,u_1,u_2;K_0)=K_2,\ I_3(x,u,u_1,u_2;K_0)=K_3, \tag{38}$$

where $K_i \in \mathbb{R}$ for $i=0,1,2,3$.

- The elimination of u_1 and u_2 from (38) in order to obtain a closed-form solution of equation (27) seems to be impossible, because both functions ψ_1 and ψ_2 and their derivatives are evaluated in $s=u_1^2-2uu_2$.
- We focus on obtaining the solution is parametric form.
- We introduce a new parameter t such that $s=s(t)$ is determined as follows:

$$s'(t)=\frac{1}{C(s(t;K_0))}, \tag{39}$$

where the prime symbol denotes derivation with respect to t.

- $s = s(t)$ satisfies

$$s'(t)^2 = 8\left(s(t)^3 - \frac{16}{3}\,\delta\, s(t) + 8K_0\right). \tag{40}$$

- $s = s(t)$ satisfies

$$s'(t)^2 = 8\left(s(t)^3 - \frac{16}{3}\,\delta\, s(t) + 8K_0\right). \tag{40}$$

- The general solution of equation (40) can be expressed as

$$s(t; t_0, K_0) = \frac{1}{2}\wp\left(t - t_0; g_2, g_3\right),$$

where $\wp(t) = \wp(t - t_0; g_2, g_3)$ denotes the Weierstrass $\wp$-function with invariants

$$g_2 = \frac{16^2}{3}\delta, \quad g_3 = -16^2 K_0. \tag{41}$$

- $s = s(t)$ satisfies

$$s'(t)^2 = 8\left(s(t)^3 - \frac{16}{3}\,\delta\, s(t) + 8K_0\right). \tag{40}$$

- The general solution of equation (40) can be expressed as

$$s(t; t_0, K_0) = \frac{1}{2}\wp\left(t - t_0; g_2, g_3\right),$$

  where $\wp(t) = \wp(t - t_0; g_2, g_3)$ denotes the Weierstrass $\wp$-function with invariants

$$g_2 = \frac{16^2}{3}\delta, \quad g_3 = -16^2 K_0. \tag{41}$$

- Let $s(t; K_0)$ be the particular solution to equation (40) corresponding to $t_0 = 0$.

- If $\psi = \psi(s; K_0)$ is a solution to the linear equation (36), then $\phi(t; K_0) = \psi(s(t; K_0); K_0)$ verifies the following Schrödinger-type equation:

$$\phi''(t; K_0) - 2\wp\left(t; \frac{16^2}{3}\delta, -16^2 K_0 \right) \phi(t; K_0) = 0. \qquad (42)$$

- If $\psi = \psi(s; K_0)$ is a solution to the linear equation (36), then $\phi(t; K_0) = \psi(s(t; K_0); K_0)$ verifies the following Schrödinger-type equation:

$$\phi''(t; K_0) - 2\wp\left(t; \frac{16^2}{3}\delta, -16^2 K_0 \right) \phi(t; K_0) = 0. \qquad (42)$$

- Therefore, if $\psi_1 = \psi_1(s; K_0)$ and $\psi_2 = \psi_2(s; K_0)$ are two linearly independent solutions to equation (36) then $\phi_1(t; K_0) = \psi_1(s(t); K_0)$ and $\phi_2(t; K_0) = \psi_2(s(t); K_0)$ is a fundamental set of solutions to equation (42).

- The implicit general solution (38) can be expressed as follows:

$$\frac{2u_1\phi_1(t;K_0)+\phi_1'(t;K_0)}{2u_1\phi_2(t)+\phi_2'(t;K_0)}=K_1,\quad \frac{2(u_1x-2u)\phi_1(t;K_0)+x\phi_1'(t;K_0)}{2(u_1x-2u)\phi_2(t;K_0)+x\phi_2'(t;K_0)}=K_2,$$

$$\frac{(2(u_1x-2u)\phi_2(t;K_0)+x\phi_2(t;K_0))^2}{4uW(\phi_1,\phi_2)(t;K_0)}=K_3. \tag{43}$$

- The implicit general solution (38) can be expressed as follows:

$$\frac{2u_1\phi_1(t;K_0)+\phi_1'(t;K_0)}{2u_1\phi_2(t)+\phi_2'(t;K_0)}=K_1,\quad \frac{2(u_1x-2u)\phi_1(t;K_0)+x\phi_1'(t;K_0)}{2(u_1x-2u)\phi_2(t;K_0)+x\phi_2'(t;K_0)}=K_2,$$

$$\frac{(2(u_1x-2u)\phi_2(t;K_0)+x\phi_2(t;K_0))^2}{4uW(\phi_1,\phi_2)(t;K_0)}=K_3. \qquad (43)$$

- We can eliminate u_1 to obtain the following parametrized general solution to equation (27):

$$\begin{aligned} x(t) &= \frac{K_3(K_1-K_2)\left(\phi_1(t;K_0)-K_2\phi_2(t;K_0)\right)}{\phi_1(t;K_0)-K_1\phi_2(t;K_0)}, \\ u(t) &= \frac{K_3(K_1-K_2)^2}{4\Big(\phi_1(t;K_0)-K_1\phi_2(t;K_0)\Big)^2}, \end{aligned}$$

where $K_i\in\mathbb{R}$ for $i=0,1,2,3$, $K_3>0$, and $K_1\neq K_2$.

Theorem

Let $\phi_1 = \phi_1(t; K_0)$ and $\phi_2 = \phi_2(t; K_0)$ be two linearly independent solutions to the one-parameter family of Schrödinger-type equations (42) such that $W(\phi_1, \phi_2)(t; K_0) = 1$. Then the exact four-parameter solution to the static Euler-Bernoulli beam equation (24) is given in parametric form by

$$
\begin{aligned}
x(t) &= K_3(K_1 - K_2)\frac{\phi_1(t;K_0) - K_2\phi_2(t;K_0)}{\phi_1(t;K_0) - K_1\phi_2(t;K_0)}, \\
y(t) &= \pm\left(\frac{K_3^{1/2}\,(\,K_1 - K_2)}{2\Big(\phi_1(t;K_0) - K_1\phi_2(t;K_0)\Big)}\right)^3,
\end{aligned}
\tag{44}
$$

where $K_i \in \mathbb{R}$ for $i = 0, 1, 2, 3$, $K_3 > 0$, and $K_1 \neq K_2$.

The case $K_0 = \pm\frac{16}{27}$

- These values of K_0 correspond precisely to the cases in which the discriminant $g_2^3 - 27g_3^2$ of the Weierstrass $\wp$-function $\wp(t) = \wp(t; g_2, g_3)$ is equal to zero.

The case $K_0 = \pm\frac{16}{27}$

- These values of K_0 correspond precisely to the cases in which the discriminant $g_2^3 - 27g_3^2$ of the Weierstrass $\wp$-function $\wp(t) = \wp(t; g_2, g_3)$ is equal to zero.
- Implies $\delta = 1$.

The case $K_0 = \pm\frac{16}{27}$

- These values of K_0 correspond precisely to the cases in which the discriminant $g_2^3 - 27g_3^2$ of the Weierstrass $\wp$-function $\wp(t) = \wp(t; g_2, g_3)$ is equal to zero.
- Implies $\delta = 1$.
- A fundamental set of solutions to the Schrödinger-type equation (42) can be obtained in terms of elementary functions.

- If $K_0 = -\frac{16}{27}$ then $\wp(t; g_2, g_3) = -\frac{8}{3} + 8\csc^2\left(2\sqrt{2}t\right)$ and it can be checked that two linearly independent solutions to the corresponding equation (42) verifying $W(\phi_1, \phi_2)(t) = 1$ become

$$\begin{aligned}
\phi_1(t) &= \frac{\sqrt{3}}{8}\cot\left(2\sqrt{2}t\right)\sin(\ \alpha(t)) - \frac{\sqrt{2}}{8}\cos(\ \alpha(t)),\\
\phi_2(t) &= \csc\left(\ 2\sqrt{2}t\right)\left(\ (3+\sqrt{6})\cos(\ \beta_1(t))\ + (3-\sqrt{6})\cos(\ \beta_2(t))\ \right),
\end{aligned} \tag{45}$$

where

$$\alpha(t) = \frac{4\sqrt{2}t - \pi}{\sqrt{6}}, \quad \beta_1(t) = \frac{4(\sqrt{2}-\sqrt{3})t - \pi}{\sqrt{6}}, \quad \beta_2(t) = \frac{4(\sqrt{2}+\sqrt{3})t - \pi}{\sqrt{6}}. \tag{46}$$

- If $K_0 = -\frac{16}{27}$ then $\wp(t; g_2, g_3) = -\frac{8}{3} + 8\csc^2\left(2\sqrt{2}t\right)$ and it can be checked that two linearly independent solutions to the corresponding equation (42) verifying $W(\phi_1, \phi_2)(t) = 1$ become

$$\begin{aligned} \phi_1(t) &= \frac{\sqrt{3}}{8}\cot\left(2\sqrt{2}t\right)\sin(\alpha(t)) - \frac{\sqrt{2}}{8}\cos(\alpha(t)), \\ \phi_2(t) &= \csc\left(2\sqrt{2}t\right)\left((3+\sqrt{6})\cos(\beta_1(t)) + (3-\sqrt{6})\cos(\beta_2(t))\right), \end{aligned} \tag{45}$$

where

$$\alpha(t) = \frac{4\sqrt{2}t - \pi}{\sqrt{6}}, \quad \beta_1(t) = \frac{4(\sqrt{2}-\sqrt{3})t - \pi}{\sqrt{6}}, \quad \beta_2(t) = \frac{4(\sqrt{2}+\sqrt{3})t - \pi}{\sqrt{6}}. \tag{46}$$

- If $K_0 = \frac{16}{27}$ then $\wp(t; g_2, g_3) = \frac{8}{3} + 8\,\mathrm{csch}^2\left(2\sqrt{2}t\right)$ and two linearly independent solutions to the corresponding equation (42) satisfying $W(\phi_1, \phi_2)(t) = 1$ become

$$\begin{aligned} \phi_1(t) &= \mathrm{csch}\left(2\sqrt{2}t\right)\left((3+\sqrt{6})\cosh(\beta_1(t)) + (3-\sqrt{6})\cosh(\beta_2(t))\right), \\ \phi_2(t) &= \frac{\sqrt{3}}{8}\coth\left(2\sqrt{2}t\right)\sinh(\alpha(t)) - \frac{\sqrt{2}}{8}\cosh(\alpha(t)), \end{aligned} \tag{47}$$

where $\alpha(t), \beta_1(t)$ and $\beta_2(t)$ are given in (46).

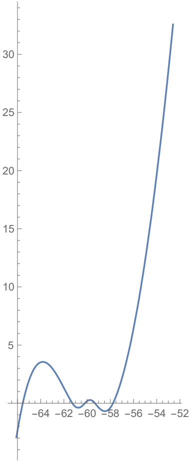

Figure: $\delta = 1, K_0 = -\frac{16}{27}, K_1 = -1, K_2 = 3, K_3 = 10,\ t \in (0.89, 6.2)$

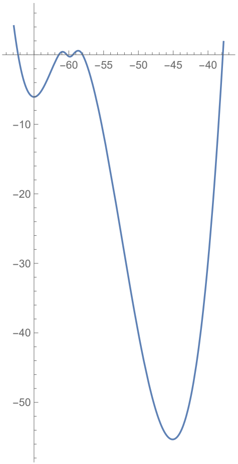

Figure: $\delta = 1, K_0 = -\frac{16}{27}, K_1 = -1, K_2 = 3, K_3 = 10,\ t \in (-11.3, -5.5)$

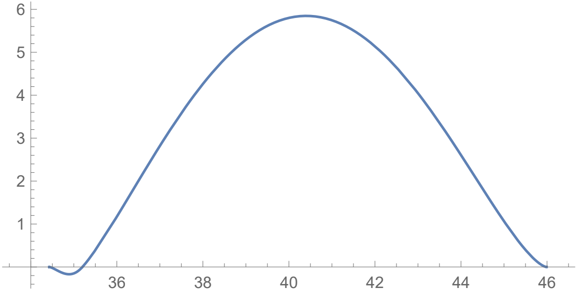

Figure: $\delta = 1, K_0 = \frac{16}{27}, K_1 = 1, K_2 = -3, K_3 = 10,\ t \in (-15, 15)$

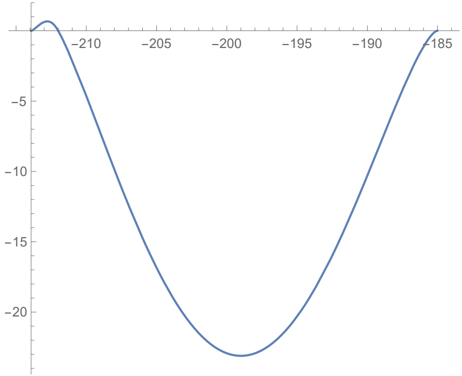

Figure: $\delta = 1, K_0 = \frac{16}{27}, K_1 = 1, K_2 = 3, K_3 = 100,\ t \in (-15, 15)$

The case $K_0 = 0$

- It can be checked that a fundamental set of solutions $\{\phi_1, \phi_2\}$ to equation (42) such that $W(\phi_1, \phi_2)(t) = 1$ is given by

$$\phi_1(t) = \sqrt{\wp(t)} \quad \text{and} \quad \phi_2(t) = \sqrt{\wp(t)}\left(\frac{6}{16^2\delta}\frac{\wp'(t)}{\wp(t)} + \frac{12}{16^2\delta}\zeta(t)\right) ,$$

where $\wp(t) = \wp(t; g_2, 0)$ and $\zeta(t) = \zeta(t; g_2, 0)$ denote the Weierstrass $\wp$ and ζ functions with parameter $g_2 = \frac{16^2}{3}\delta$.

The case $K_0 = 0$

- It can be checked that a fundamental set of solutions $\{\phi_1, \phi_2\}$ to equation (42) such that $W(\phi_1, \phi_2)(t) = 1$ is given by

$$\phi_1(t) = \sqrt{\wp(t)} \quad \text{and} \quad \phi_2(t) = \sqrt{\wp(t)} \left(\frac{6}{16^2\delta} \frac{\wp'(t)}{\wp(t)} + \frac{12}{16^2\delta}\zeta(t) \right) ,$$

  where $\wp(t) = \wp(t; g_2, 0)$ and $\zeta(t) = \zeta(t; g_2, 0)$ denote the Weierstrass $\wp$ and ζ functions with parameter $g_2 = \frac{16^2}{3}\delta$.

- The following three-parameter family of solutions to equation (24) is obtained:

$$x(t) = K_3(K_1 - K_2) \left(\frac{16^2\delta\wp(t) - K_2 \left(6\wp'(t) + 12\wp(t)\zeta(t) \right)}{16^2\delta\wp(t) - K_1 \left(6\wp'(t) + 12\wp(t)\zeta(t) \right)} \right),$$

$$y(t) = \pm \left(\frac{k_3^{1/2}(K_1 - K_2)16^2\delta\wp(t)}{2\sqrt{\wp(t)} \left(16^2\delta\wp(t) - K_1 \left(6\wp'(t) + 12\zeta(t)\wp(t) \right) \right)} \right)^3,$$

  where $K_i \in \mathbb{R}$ for $i = 1, 2, 3,\ K_3 > 0,\ K_1 \neq K_2$.

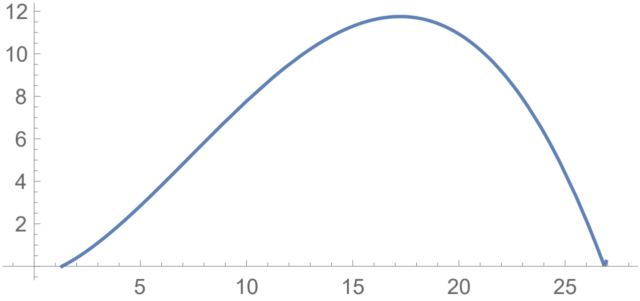

Figure: $\delta = 1, K_0 = 0, K_1 = 0.3, K_2 = -1, K_3 = 1$

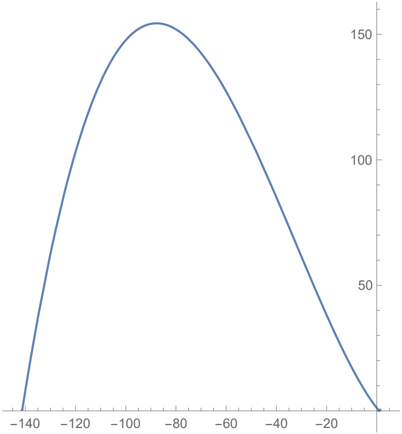

Figure: $\delta = 1, K_0 = 0, K_1 = -0.3, K_2 = -10, K_3 = 0.1$

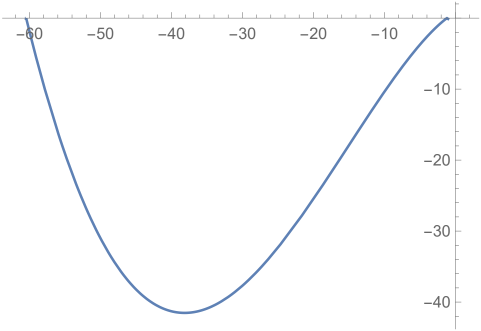

Figure: $\delta = 1, K_0 = 0, K_1 = -0.3, K_2 = 3, K_3 = 0.3$

The case $K_0 \neq \pm\frac{16}{27}$ and $K_0 \neq 0$

- A fundamental set of solutions $\{\phi_1, \phi_2\}$ to equation (42) such that $W(\phi_1, \phi_2)(t) = 1$ is determined by

$$\phi_1(t) = \wp(t)\exp(-t\zeta(\gamma))\frac{\sigma(t)\sigma(\gamma)}{\sigma(t-\gamma)}, \quad \phi_2(t) = \frac{1}{\wp'(\gamma)}\exp(t\zeta(\gamma))\frac{\sigma(t-\gamma)}{\sigma(t)\sigma(\gamma)},$$

where $\wp(t) = \wp(t; g_1, g_2)$, $\zeta(t) = \zeta(t; g_1, g_2)$, $\sigma(t) = \sigma(t; g_1, g_2)$ stand for the Weierstrass $\wp$-function, ζ-function and σ-function, respectively, with parameters

$$g_1 = \frac{16^2}{3}\delta \quad \text{and} \quad g_2 = -16^2 K_0,$$

and the value γ is such that $\wp(\gamma) = 0$.

Theorem

The four-parameter general solution to the Euler-Bernoulli beam equation

$$y_4 = \delta y^{-5/3}, \qquad \delta = \pm 1,$$

is given in parametric form through

$$x(t) = K_3(K_1 - K_2)\left(\frac{\sigma(\gamma)^2\sigma(t)^2\wp(t)\wp'(\gamma)exp(-t\zeta(\gamma)) - K_2 exp(t\zeta(\gamma))\sigma(t-\gamma)^2}{\sigma(\gamma)^2\sigma(t)^2\wp(t)\wp'(\gamma)exp(-t\zeta(\gamma)) - K_1 exp(t\zeta(\gamma))\sigma(t-\gamma)^2}\right),$$

$$y(t) = \pm\left(\frac{K_3^{1/2}(K_1 - K_2)\sigma(\gamma)\sigma(t)\sigma(\gamma - t)\wp'(\gamma)}{2\left(\sigma(\gamma)^2\sigma(t)^2\wp(t)\wp'(\gamma)exp(-t\zeta(\gamma)) - K_1 exp(t\zeta(\gamma))\sigma(t-\gamma)^2\right)}\right)^3,$$

where $K_1, K_2, K_3 \in \mathbb{R}$, $K_1 \neq K_2$, $K_3 > 0$, $\wp(t) = \wp(t; g_1, g_2)$, $\zeta(t) = \zeta(t; g_1, g_2)$, $\sigma(t) = \sigma(t; g_1, g_2)$ *stand for the Weierstrass* $\wp$*-function,* ζ*-function and* σ*-function, respectively, with parameters* $g_1 = \frac{16^2}{3}\delta$ *and* $g_2 = -16^2 K_0$, $K_0 \in \mathbb{R} \setminus \left\{\pm\frac{16}{27}, 0\right\}$, *and the value* γ *is such that* $\wp(\gamma) = 0$.

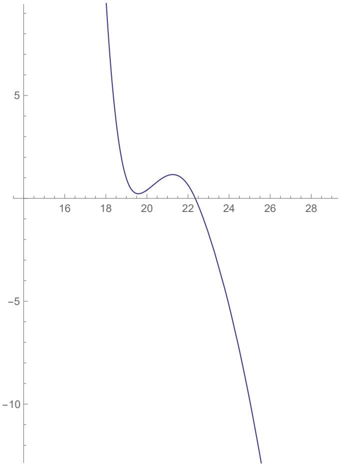

Figure: $\delta = 1, K_0 = 1, K_1 = -1, K_2 = -3, K_3 = 10,\ t \in (-1.6, 1)$

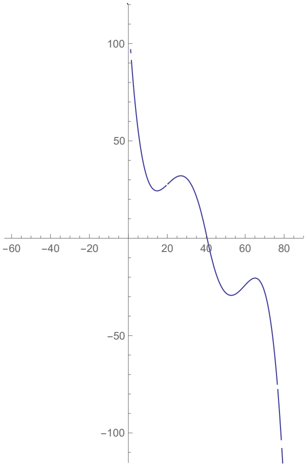

Figure: $\delta = 1, K_0 = 1, K_1 = -1, K_2 = -3, K_3 = 10,\ t \in (1, 4)$

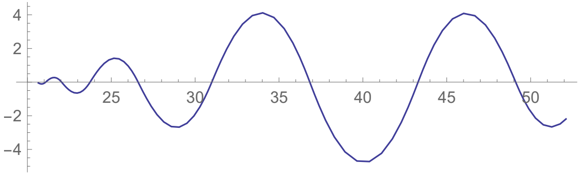

Figure: $\delta = 1, K_0 = 0.1, K_1 = -1, K_2 = -3, K_3 = 10,\ t \in (-100, 100)$

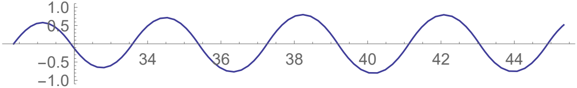

Figure: $\delta = 1, K_0 = 0.01, K_1 = -1, K_2 = -3, K_3 = 10,\ t \in (-100, 100)$

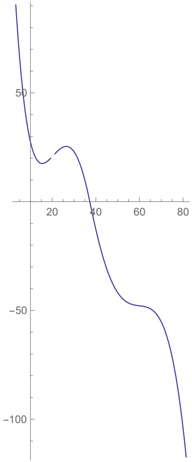

Figure: $\delta = 1, K_0 = 1.1, K_1 = -1, K_2 = -3, K_3 = 10,\ t \in (1.1, 3.6)$

A singular two-parameter family of solutions

- We study the singular case in which the function $C(s; K_0)^2$ is not defined.

A singular two-parameter family of solutions

- We study the singular case in which the function $C(s; K_0)^2$ is not defined.
- That situation occurs if $s = s_0(K_0)$, being $s_0(K_0)$ one of the roots of the polynomial $s^3 - \frac{16}{3}\delta s + 8K_0 = 0$.

A singular two-parameter family of solutions

- We study the singular case in which the function $C(s; K_0)^2$ is not defined.
- That situation occurs if $s = s_0(K_0)$, being $s_0(K_0)$ one of the roots of the polynomial $s^3 - \frac{16}{3}\,\delta\, s + 8K_0 = 0$.
- According to the value of s, $s = s_0(K_0)$ yields the following second-order equation

$$u_1^2 - 2uu_2 = s_0(K_0).$$

A singular two-parameter family of solutions

- We study the singular case in which the function $C(s;K_0)^2$ is not defined.
- That situation occurs if $s = s_0(K_0)$, being $s_0(K_0)$ one of the roots of the polynomial $s^3 - \frac{16}{3}\,\delta\, s + 8K_0 = 0$.
- According to the value of s, $s = s_0(K_0)$ yields the following second-order equation

$$u_1^2 - 2uu_2 = s_0(K_0).$$

- The solutions $u = f(x)$ are of the form

$$f(x) = ax^2 + bx + c, \qquad (48)$$

  where the constants a, b and c satisfy the condition $s_0(K_0) = b^2 - 4ac$.

- It can be checked that (48) satisfies the equation (27) if and only if

$$3(b^2 - 4ac)^2 - \frac{16}{3}\delta = 0. \tag{49}$$

- It can be checked that (48) satisfies the equation (27) if and only if

$$3(b^2 - 4ac)^2 - \frac{16}{3}\delta = 0. \tag{49}$$

- Solutions (48) verifying (49) correspond to solutions of (24) of the form

$$y(x) = \pm(ax^2 + bx + c)^{3/2}, \tag{50}$$

when the constants a, b and c satisfy (49).